May and Lie-May spectral sequences in chromatic stable homotopy theory and $p$-adic geometry, part 2.

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Outline

The task at hand.

LMSS differentials.

Functoriality in the choice of base field.

The $A$-height 1 LMSS.

$A$-height 2.
Let $C$ be the filtered cochain complex with nontrivial cohomology which is a tensor factor in the filtered cochain complex which gives us the LMSS.
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We want to compute the spectral sequence associated to $C^\cdot$, since $H^*(C^\cdot)$ is the $E_2$-term of the May SS converging to the group cohomology of the height $n$ Morava stabilizer group/algebra.

Tools we will use to accomplish this:
1. An understanding of the differentials in the spectral sequence.
2. Functoriality in the spectral sequence in choice of base field.
3. A computation of the $E_2$-term (accomplished by Ravenel).
4. A characteristic zero deformation of $C^\cdot$.
5. Willingness to program a computer.
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Let $k$ be a field of characteristic $p$ and let $A$ be a primitively generated cocommutative Hopf algebra over $k$. There are too many primitives in $UPA$, and this makes $H^*(UPA; k)$ too big; the differentials in the Lie-May SS correct for this. Let’s see how and why with a small example.
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Let $A \cong \mathbb{F}_2[x]/x^4$ with $x$ primitive. Then $PA \cong \mathbb{F}_2\{x, x^2\}$ with $\xi(x) = x^2$ and $\xi(x^2) = 0$. So $UPA \cong \mathbb{F}_2[x_0, x_1]$ and $H^*(UPA; \mathbb{F}_2) \cong E(h_0, h_1)$, with $|h_0| = |h_1| = 1$. 
The LMSS for $A \cong \mathbb{F}_2[x]/x^4$.

\[
E(h_0, h_1) \otimes \mathbb{F}_2[b_0, b_1] \Rightarrow H^*(VPA; \mathbb{F}_2).
\]
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$$E(h_0, h_1) \otimes \mathbb{F}_2[b_0, b_1] \Rightarrow H^*(VPA; \mathbb{F}_2).$$

The linear dual of $A$ is $A^* \cong \mathbb{F}_2[t_0, t_1]/t_0^2, t_1^2$ with $t_0$ primitive and $
\Delta(t_1) = t_1 \otimes 1 + t_0 \otimes t_0 + 1 \otimes t_1$. Cocycle representatives in the cobar complex: $t_0$ represents $h_0$, $t_0 \otimes t_0$ represents $b_0$, $t_1$ represents $h_1$, $t_1 \otimes t_1 + t_0 \otimes t_0 t_1 + t_0 t_1 \otimes t_0$ represents $b_1$.
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In general the class in $k[PA^+]$ associated to an element $x \in PA^+$ is the transpotent on $x$:

$$\frac{1}{p} ( (x \otimes 1 + 1 \otimes x)^p - x^p \otimes 1 - 1 \otimes x^p ).$$
A differential.

In the 2-primary case, transpotents are just cup product squares; $b_1$ is represented by the transpotent $t_1 \otimes t_1$ of $t_1$, and we add the terms $t_0 \otimes t_0 t_1 + t_0 t_1 \otimes t_0$ of lower Lie-May degree to get a cocycle.
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So $E_{\infty,*} \cong H^*(A; \mathbb{F}_2) \cong E(h_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[b_1]$. 
Exotic multiplicative extensions.

Notice that if we did this for $\mathbb{F}_2[x]/x^2$ instead of $\mathbb{F}_2[x]/x^4$, there would be no differentials, but the LMSS $E_{\infty,*}^*$-term would look like $E(h) \otimes_{\mathbb{F}_2} \mathbb{F}_2[b]$, when we know (from direct computation) that the answer should really be $\mathbb{F}_2[h]$. 
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So there is an exotic multiplicative extension: $h^2$ appears to be zero in the spectral sequence, but it really is nonzero in the cohomology of $A$. 
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So there is an exotic multiplicative extension: $h^2$ appears to be zero in the spectral sequence, but it really is nonzero in the cohomology of $A$.

This is not an isolated phenomenon but in fact happens constantly in the LMSS at higher heights, at more or less all primes. Exotic multiplicative extensions can be resolved by computing cocycle representatives for classes, or by trickier methods (which we’ll get to).
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A-height 2.
Base field.

Everything we have said so far is functorial in the choice of base field. What does this mean?
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If $K/\mathbb{Q}_p$ is a finite field extension, let $A$ denote the ring of integers in $K$. Then one can consider formal groups $F/F_p$ equipped with a choice of extension of the ring map $\mathbb{Z} \to \text{End}(F)$ to a ring map $A \to \text{End}(F)$—that is, one can consider formal $A$-modules.
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If $[K : \mathbb{Q}_p] = d$ and $F/\mathbb{F}_p$ is a formal group law of $p$-height $n$, then $F$ admits a formal $A$-module structure iff $d|n$. 
If $F$ is a formal $A$-module with $p$-height $n$, and $[K : \mathbb{Q}_p] = d$, we say that $F$ has $A$-height $n/d$. (There is also an intrinsic definition, in terms of a $p$-series, of $A$-height which generalizes $p$-height.)
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Let $F^A_{n/d}$ be a formal $A$-module of $A$-height $n/d$ and let $F_n$ be its underlying formal group (of $p$-height $n$). Then we have a commutative diagram

$$
\begin{array}{ccc}
\text{Aut}(F^A_{n/d}) & \longrightarrow & \text{Aut}(F_n) \\
\downarrow \cong & & \downarrow \cong \\
\text{Syl}_p(O_{D_{d/n,K}}^\times) & \longrightarrow & \text{Syl}_p(O_{D_{1/n,\mathbb{Q}_p}}^\times).
\end{array}
$$
Some tangential remarks.

(One can generalize this to higher-dimensional formal modules and realize all finite-rank central division algebras as endomorphism rings of formal modules; the functoriality of the Brauer group is a decategorification of the functoriality of taking these endomorphism algebras!)
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(In fact these inclusions of pro-$p$-groups are the maps induced on geometric points by the morphism from the moduli stack of formal $A$-modules to the moduli stack of formal groups.)
We’re going to use the restriction map induced in cohomology

\[ H^*(\text{Syl}_p(O^\times_{D_{1/n},\mathbb{Q}_p}); \mathbb{F}_p) \to H^*(\text{Syl}_p(O^\times_{D_{d/n},K}); \mathbb{F}_p) \]

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to help us compute LMSS differentials.

Now the continuous linear dual of \( \mathbb{F}_p^n[\text{Syl}_p(\mathcal{O}_{D_{1/n},\mathbb{Q}_p}^\times)] \) is isomorphic to \( \mathbb{F}_p^n[t_1, t_2, \ldots ]/(t_i^{p^n} - t_i) \), so the continuous linear dual of \( \mathbb{F}_p^n[\text{Syl}_p(\mathcal{O}_{D_{1/n},\mathbb{Q}_p}^\times)] \) should be a quotient of this Hopf algebra.
If $ef | n$ and $K = \mathbb{Q}_p[\sqrt[ef]{\alpha p}]$, where $\alpha \in \mathcal{W}(\mathbb{F}_{p^f})^\times$ is not in the image of $\mathcal{W}(\mathbb{F}_{p^{f-1}})^\times \hookrightarrow \mathcal{W}(\mathbb{F}_{p^f})^\times$, and let $c$ be a primitive $\frac{p^e - 1}{p - 1}$th root of the mod $p$ reduction of $\alpha$. Then, as a quotient of the Morava stabilizer algebra, the continuous $\mathbb{F}_p$-linear dual of $\mathbb{F}_p[\text{Syl}_p(\mathcal{O}_{D_{ef/n,K}}^\times)]$ is

$$
\mathbb{F}_p[\mathbb{t}_f, \mathbb{t}_{2f}, \ldots]/(\mathbb{t}_i^{p^n/e} - c^{p^i - 1}\mathbb{t}_i).
$$

(This includes all extensions of $\mathbb{Q}_p$ which aren’t wildly ramified, as well as some wildly ramified ones, e.g. $\mathbb{Q}_2[\zeta_8]/\mathbb{Q}_2$, which is totally ramified of degree 4 and detects the Kervaire invariant class in the Adams $E_2$.)
If \( ef \mid n \) and \( K = \mathbb{Q}_p[\sqrt[p]{\alpha}] \), where \( \alpha \in \mathcal{W}(\mathbb{F}_{p^f})^\times \) is not in the image of \( \mathcal{W}(\mathbb{F}_{p^{f-1}})^\times \hookrightarrow \mathcal{W}(\mathbb{F}_{p^f})^\times \), and let \( c \) be a primitive \( \frac{p^e-1}{p-1} \)th root of the mod \( p \) reduction of \( \alpha \). Then, as a quotient of the Morava stabilizer algebra, the continuous \( \mathbb{F}_{p^{fn}} \)-linear dual of

\[
\mathbb{F}_{p^{fn}}[\text{Syl}_p(\mathcal{O}_{D_{ef/n,K}}^\times)]
\]

is

\[
\mathbb{F}_{p^{fn}}[t_f, t_{2f}, \ldots]/(t_i^{p^n/e} - c^{p^i-1}t_i).
\]

(This includes all extensions of \( \mathbb{Q}_p \) which aren’t wildly ramified, as well as some wildly ramified ones, e.g. \( \mathbb{Q}_2[\zeta_8]/\mathbb{Q}_2 \), which is totally ramified of degree 4 and detects the Kervaire invariant class in the Adams \( E_2 \).)
Functoriality of the MSS.

The Ravenel filtration induces a filtration on the continuous linear dual of $S^A(n/ef) \cong \mathbb{F}_p^n[Syl_p(O_{D_{ef/n,K}}^\times)]$ and hence a morphism of May spectral sequences

\[
\begin{array}{c}
H^*(E^0 S(n)^*; \mathbb{F}_{p^n}) \rightarrow H^*(Syl_p(O_{D_{1/n,Q_p}}^\times); \mathbb{F}_{p^n})
\end{array}
\]

\[
\begin{array}{c}
\downarrow \quad \downarrow \\
H^*(E^0 S^A(n/ef)^*; \mathbb{F}_{p^n}) \rightarrow H^*(Syl_p(O_{D_{ef/n,K}}^\times); \mathbb{F}_{p^n}).
\end{array}
\]
Functoriality of the LMSS.

And we have a morphism of Lie-May spectral sequences:

$$
\begin{align*}
H^*(UPE^0 S(n)^*; \mathbb{F}_{p^n}) \otimes_{\mathbb{F}_{p^n}} \mathbb{F}_{p^n}[PE^0 S(n)^*] & \rightarrow H^*(E^0 S(n)^*; \mathbb{F}_{p^n}) \\
\downarrow & \downarrow \\
H^*(UPE^0 S^A(n/ef)^*; \mathbb{F}_{p^n}) & \rightarrow H^*(E^0 S^A(n/ef)^*; \mathbb{F}_{p^n})
\end{align*}
$$

Now the filtered cochain complex defining the top spectral sequence splits as a tensor product $C\cdot \otimes D\cdot$ while the filtered cochain complex defining the bottom spectral sequence splits as a tensor product $C\cdot A \otimes D\cdot A$, and these splittings are functorial in $K/\mathbb{Q}_p$, i.e., the image of $C\cdot$ lands in $C\cdot A$. The factors $D\cdot$ and $D\cdot A$ are cohomologically trivial.
Functoriality of the LMSS.

And we have a morphism of Lie-May spectral sequences:

\[
H^\ast(\text{UPE}^0 S(n)^*; \mathbb{F}_{p^n}) \otimes_{\mathbb{F}_{p^n}} \mathbb{F}_{p^n}[P \text{E}^0 S(n)^*] \to H^\ast(E^0 S(n)^*; \mathbb{F}_{p^n})
\]

\[
\downarrow \quad \downarrow
\]

\[
H^\ast(\text{UPE}^0 S^A(n/ef)^*; \mathbb{F}_{p^n}) \to H^\ast(E^0 S^A(n/ef)^*; \mathbb{F}_{p^n})
\]

Now the filtered cochain complex defining the top spectral sequence splits as a tensor product \( C \cdot \otimes D \cdot \) while the filtered cochain complex defining the bottom spectral sequence splits as a tensor product \( C_A \cdot \otimes D_A \cdot \), and these splittings are functorial in \( K/\mathbb{Q}_p \), i.e., the image of \( C \) lands in \( C_A \cdot \). The factors \( D \cdot \) and \( D_A \cdot \) are cohomologically trivial.
Here are the cohomologically nontrivial factors in the LMSS $E_2$-terms:

$$H^*(L(n, \frac{pn}{p-1}); \mathbb{F}_{p^f}) \otimes_{\mathbb{F}_{p^f}} \mathbb{F}_{p^f} \left[ \left\{ b_{i,j} : 1 \leq i \leq \frac{n}{p-1}, 0 \leq j \leq n-1 \right\} \right]$$

$$\quad \to H^*(L^A(n/d, \frac{pn}{p-1}); \mathbb{F}_{p^f})$$

$$\otimes_{\mathbb{F}_{p^f}} \mathbb{F}_{p^f} \left[ \left\{ b_{i,j} : f \mid i, i \leq \frac{n}{p-1}, 0 \leq j \leq \frac{n}{e} - 1 \right\} \right]$$
Here are the cohomologically nontrivial factors in the LMSS $E_2$-terms:

$$H^\ast(L(n, \frac{pn}{p-1}); \mathbb{F}_{p\mathfrak{fn}}) \otimes_{\mathbb{F}_{p\mathfrak{fn}}} \mathbb{F}_{p\mathfrak{fn}} \left[ \left\{ b_{i,j} : 1 \leq i \leq \frac{n}{p-1}, 0 \leq j \leq n - 1 \right\} \right]$$

$$\rightarrow H^\ast(L^A(n/d, \frac{pn}{p-1}); \mathbb{F}_{p\mathfrak{fn}})$$

$$\otimes_{\mathbb{F}_{p\mathfrak{fn}}} \mathbb{F}_{p\mathfrak{fn}} \left[ \left\{ b_{i,j} : f | i, i \leq \frac{n}{p-1}, 0 \leq j \leq \frac{n}{e} - 1 \right\} \right]$$

The unrestricted Lie algebras $L(n, \frac{pn}{p-1})$ and $L^A(n/d, \frac{pn}{p-1})$ each have composition series of length $\frac{pn}{p-1}$ where the composition factors are abelian; so their cohomologies can be computed via $\frac{pn}{p-1} - 1$ Hochschild-Serre spectral sequences (and isn’t hard).
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$A$-height 2.
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$$H^*(L^A(1, \frac{pe}{p-1}); \mathbb{F}_{p^e}) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e} \left[ \left\{ b_{i,0} : 1 \leq i \leq \frac{e}{p-1} \right\} \right]$$

$$\cong E \left( \left\{ h_{i,0} : 1 \leq \frac{pe}{p-1} \right\} \right) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e} \left[ \left\{ b_{i,0} : 1 \leq i \leq \frac{e}{p-1} \right\} \right]$$

Differentials:

1. $dh_{pe,0}^1$ hits a nonzero scalar times $b_{e-p}^1$, if $K$ does not contain a primitive $p$th root of unity, and $dh_{pe,0}^1 = 0$ if $K$ contains a primitive $p$th root of unity.

2. no further differentials are possible.
Let $K = \mathbb{Q}_p[\sqrt[pe]{\alpha p}]$ with $\alpha \in \hat{\mathbb{Z}}_p^\times$.

\[
H^*(L^A(1, \frac{pe}{p-1}); \mathbb{F}_{p^e}) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e} \left[ \left\{ b_{i,0} : 1 \leq i \leq \frac{e}{p-1} \right\} \right]
\]

\[
\cong E(\left\{ h_{i,0} : 1 \leq i \leq \frac{pe}{p-1} \right\}) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e} \left[ \left\{ b_{i,0} : 1 \leq i \leq \frac{e}{p-1} \right\} \right]
\]

Differentials:

1. $dh_{p^e,0}$ hits a nonzero scalar times $b_{\ell,0}$, if $\ell < \frac{e}{p-1}$

2. $dh_{\frac{pe}{p-1},0}$ hits a nonzero scalar times $b_{\frac{pe}{p-1},0}$ if $K$ does not contain a primitive $p$th root of unity, and $dh_{\frac{pe}{p-1},0} = 0$ if $K$ contains a primitive $p$th root of unity.

3. no further differentials are possible.
So $H^\ast(E^0 S^A(1)^\ast; \mathbb{F}_{p^e})$ when $p > 2$ is:

$$E \left( \left\{ h_{i,0} : 1 \leq i < \frac{pe}{p-1}, p \nmid i \right\} \right)$$

if $K$ has no prim. $p$th root of 1, and

$$E \left( \left\{ h_{i,0} : 1 \leq i < \frac{pe}{p-1}, p \nmid i \right\} \right) \otimes_{\mathbb{F}_{p^e}} E(h_{\frac{pe}{p-1},0}) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e}[b_{\frac{e}{p-1},0}]$$

if $K$ has a prim. $p$th root of 1. When $p = 2$ the only difference is that $h_{e,0}^2 = b_{e,0}$ instead of being zero.
Now we can do $B$-height 2 computations, for all number rings $B$ of the above type, and all primes, by using the restriction maps to the $A$-height 1 spectral sequences where $K(A)/K(B)$ is quadratic; then we can do $C$-height 4 computations by using the restriction maps to the $B$-height 2 spectral sequences where $K(B)/K(C)$ is quadratic; etc. The restriction maps allow computation of LMSS differentials and exotic multiplicative extensions when finding cocycle representatives in the cobar complex is prohibitively hard.
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$A$-height 2.
First extension in composition series for $L^A(2, \frac{2ep}{p-1})$, at all primes:

$$
1 \to L^A(2, 1) \to L^A(2, 2) \to E(h_{2,0}, h_{2,1}) \to 1,
$$

with $L^A(2, 1) \cong E(h_{1,0}, h_{1,1})$. 
First extension in composition series for $L^A(2, \frac{2ep}{p-1})$, at all primes:

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with $L^A(2,1) \cong E(h_{1,0}, h_{1,1})$.

If $p > 2$ then let $\zeta_2 = c^{p-1}h_{2,0} + h_{2,1}$ and let $\eta_2 = -c^{p-1}h_{2,0} + h_{2,1}$. The HSSS splits as a tensor product with one tensor factor having cohomology $E(\zeta_2)$ and the other having differential $d\eta_2 = -2c^{p-1}h_{10}h_{11}$. 
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(Draw HSSS picture on board.)
Next extension in composition series for $L^A(2, \frac{2ep}{p-1})$, when $p > 2$:

$$1 \to L^A(2, 2) \to L^A(2, 3) \to E(h_{2,0}, h_{2,1}) \to 1,$$

with

$\text{H}^*(L^A(2, 2); \mathbb{F}_{p^e}) \cong \mathbb{F}_{p^e}\{1, h_{1,0}, h_{1,1}, \eta_2 h_{1,0}, \eta_2 h_{1,1}, \eta_2 h_{1,0} h_{1,1}\}.$
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Suppose $3 \leq \frac{2pe}{p-1} < 4$. Then we have the LMSS:

$$H^*(L^A(2, 3); \mathbb{F}_{p^e}) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e}[b_{1,0}, b_{1,1}] \Rightarrow H^*(E^0 S^A(2)^*; \mathbb{F}_{p^e}).$$
Suppose \( 3 \leq \frac{2p^e}{p-1} < 4 \). Then we have the LMSS:
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dh_{1,0}h_{3,0} = h_{1,0}(h_{1,0} \eta_2) - h_{1,0}b_{1,1}.
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H^*(L^A(2, 3); \mathbb{F}_{p^e}) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e}[b_1, 0, b_1, 1] \Rightarrow H^*(E^0 S^A(2)^*; \mathbb{F}_{p^e}).
\]

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\]

Compute the exotic multiplicative extension \( h_{1,0}(h_{1,0}\eta_2) \):
Suppose $3 \leq \frac{2p_e}{p-1} < 4$. Then we have the LMSS:

$$H^*(L^A(2, 3); \mathbb{F}_{p^e}) \otimes_{\mathbb{F}_{p^e}} \mathbb{F}_{p^e}[b_{1,0}, b_{1,1}] \Rightarrow H^*(E^0 S^A(2)^*; \mathbb{F}_{p^e}).$$

$$dh_{1,0}h_{3,0} = h_{1,0}(h_{1,0} \eta_2) - h_{1,0}b_{1,1}.$$ 

Compute the exotic multiplicative extension $h_{1,0}(h_{1,0} \eta_2)$:

$$(\eta_2 h_{10})h_{1,0} = t_2 \otimes t_1 \otimes t_1 - t_2^p \otimes t_1 \otimes t_1 - t_1^{p+1} \otimes t_1 \otimes t_1$$

$$+ 1 \frac{1}{2} t_1^p \otimes t_1^2 \otimes t_1$$

$$= h_{1,1}b_{1,0}.$$