Adams spectral sequences, twisted deformation theory, and nonabelian higher-order Hochschild cohomology.

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Outline

Adams spectral sequences.

Stable representation semirings.

Some important example cases.

Classical (untwisted) deformation theory of modules.

Hochschild (co)homology.
Let $E$ be an $E_\infty$-ring spectrum, $M$ a spectrum. There exists a spectral sequence

$$E_2^{s,t} \simeq \text{Ext}_{E^*E}^s(E^*(M), \Sigma^t \pi_*(E)) \Rightarrow \pi_{t-s}(\hat{M}_E),$$

the $E$-Adams spectral sequence. (Modulo convergence issues. Everything is okay if $M, E$ are both connective.)
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Under appropriate conditions, these objects get more identifiable; e.g., if $E = HF_p$ and $A$ is the mod $p$ Steenrod algebra, it looks like

$$E_2^{s,t} \cong \text{Ext}^s_A(H^*(M; F_p), F_p) \Rightarrow \tilde{\pi}_{t-s}(\hat{M}).$$
Under some conditions this picture simplifies even further. Let $E = H\mathbb{F}_2$, and let $A(n)$ be the subalgebra of $A$ generated by $Sq^1, \ldots, Sq^{2^n}$. Let $E(n)$ be the (exterior) subalgebra of $A$ generated by the first $n$ Quillen primitives $Q_0, Q_1, \ldots, Q_{n-1}$. 
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Then we have isomorphisms

\[
H^*(HZ; \mathbb{F}_2) = A \otimes_{A(0)} \mathbb{F}_2, \\
H^*(ku; \mathbb{F}_2) = A \otimes_{E(1)} \mathbb{F}_2, \\
H^*(ko; \mathbb{F}_2) = A \otimes_{A(1)} \mathbb{F}_2, \\
H^*(BP\langle n \rangle; \mathbb{F}_2) = A \otimes_{E(n)} \mathbb{F}_2, \\
H^*(tmf; \mathbb{F}_2) = A \otimes_{A(2)} \mathbb{F}_2.
\]
As a result of these isomorphisms we can use a change of rings

$$\text{Ext}^*_A(A \otimes_B M, \mathbb{F}_2) \cong \text{Ext}^*_B(M, \mathbb{F}_2)$$

to reduce the computation of the cohomology of the Adams
\(E_2\)-term for computing 2-complete integral homology, 2-complete
real or complex \(K\)-theory, 2-complete \(\text{tmf}\), etc., to the computation
of cohomology over \(B\). Here \(B\) is any of the various
sub-Hopf-algebras of \(A\) mentioned above: \(E(n)\) or \(A(n)\).
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Let $k$ be a field, $A$ a graded, co-commutative Hopf algebra over $k$. Then we let $\text{Stab}(A)$ be the category of graded $A$-modules of finite dimension over $k$, modulo the following equivalence relation on hom-sets: two maps $f, g : M \to N$ of $A$-modules are declared stably equivalent if $f - g$ factors through a projective $A$-module.

If $A$ is connective and either finite-dimensional over $k$ or a sequential colimit of finite-dimensional Hopf algebras over $k$, then the higher $\text{Ext}_A$ groups with coefficients in $M$ depend only on the isomorphism class of $M$ in $\text{Stab}(A)$. So we can think of $\text{Ext}_i^A(-, k)$, for $i > 0$, as a functor on $\text{Stab}(A)$. 
Stable representation semirings

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Furthermore, $\text{Stab}(A)$ has a direct sum operation and a tensor product $\otimes_k$ operation, inherited from these operations on $A$-modules. The direct sum operation is respected by the higher $\text{Ext}$ groups:

$$\text{Ext}_A^i(M \oplus N, k) \cong \text{Ext}_A^i(M, k) \oplus \text{Ext}_A^i(N, k),$$

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Let’s write $\text{StabRep}(A)$ for the semiring of isomorphism classes in $\text{Stab}(A)$. 
So one way to get a good picture of the category of 2-complete finite spectra is to regard \( \text{StabRep}(E(n)) \) as an kind of approximation, “seen through the eyes of \( BP\langle n \rangle \),” to the category of finite spectra. If you compute \( \text{Ext}^*_E(n)(M, \mathbb{F}_2) \) for a set of additive generators \( M \in \text{StabRep}(E(n)) \), then you know all the possible Adams \( E_2 \)-terms converging to \( \hat{\pi}_*(BP\langle n \rangle \wedge X)_2 \), for finite spectra \( X \). If you can compute \( \text{Ext}^*_E(n)(M, \mathbb{F}_2) \) for at least a set of multiplicative generators \( M \) of \( \text{StabRep}(E(n)) \), you’re still in extremely good shape. Similarly with \( ko \) and \( A(1) \) or \( tmf \) and \( A(2) \) in place of \( BP\langle n \rangle \) and \( E(n) \).
Or, for a commutative ring spectrum $E$, one can define and try to compute $\text{StabRep}(E^*E)$, and this is a kind of approximation to the category of finite spectra as well; again, if you can compute $\text{Ext}_{E^*E}^*(M, \pi_*(E))$ for generators $M$ of $\text{StabRep}(E^*E)$, you know all the $E$-Adams $E_2$-terms converging to $\pi_*(\hat{X}_E)$ for finite spectra $X$. (You still don’t know exactly which $E^*E$-modules actually are the $E$-cohomology of spectra; that’s another problem entirely, the realization problem.)
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Classical (untwisted) deformation theory of modules.

Hochschild (co)homology.
Finite-dimensional cocommutative Hopf algebras over a field can be written canonically as extensions of etale Hopf algebras by connective Hopf algebras:

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As a consequence, connective graded Hopf algebras and etale Hopf algebras (i.e., twists of finite group rings) are the important families of Hopf algebras to try to understand.
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The group ring example which is most important to topology, which the rest of this talk won’t be about, is as follows: when \( F \) is a height \( n \) formal group over \( \overline{\mathbb{F}}_p \), one has an \( E_\infty \)-ring spectrum \( E(F) \) equipped with an action of \( \text{Aut}(F) \) by \( E_\infty \)-ring spectrum automorphisms.
Evaluated on a finite spectrum $X$, $E(F)^*(X)$ is a $p$-adic Banach representation of $\text{Aut}(F)$. Some open conjectures in $p$-adic Hodge theory state that specifying such a $p$-adic representation of $\text{Aut}(F)$ should be equivalent to specifying a $p$-adic representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. (The numerical data attached to such a representation (the local $L$- and $\epsilon$-factors) should be closely related, through denominators of their special values, to the orders of groups appearing in certain spectral sequences converging to $\pi_*(X)$ or $\pi_*(L_{K(n)}X)$; this is what “topological Langlands correspondences” are supposed to be about.)
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In order to actually prove such a thing, one wants to compute

$$\text{StabRep}(\overline{\mathbb{F}}_p[\text{Aut}(F)]) \cong \text{StabRep}(E(F)^*E(F)/(p, v_1, \ldots, v_{n-1}, v_n-1)),$$

and then “deform” this computation to

$$\text{StabRep}(E(F)^*E(F)/(p, v_1, \ldots, v_{n-1})), \text{ then to }$$

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$\text{StabRep}(E(F)^*E(F)/(p, v_1, \ldots, v_{n-2}))$, and so on; this mimics how we do the computations in cohomology. And at the first stage one can compare one’s results with known constructions of mod $p$ Langlands correspondences, then try to understand how the correspondences change under deformation.
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We begin with a more tractable case, in which we try to compute $\text{StabRep}(A)$ for $A$ connective, rather than a group ring. Specifically, let $A$ be a sub-Hopf-algebra of the Steenrod algebra. (This case is still hard enough that Margolis writes that it “is a very difficult problem in general.”)
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The stable representation semirings on the ends are computable by “elbow grease”:

\[
\text{StabRep}(E(Sq^2)) \cong \mathbb{N}[\Sigma^{\pm 1}],
\]

\[
\text{Stab Rep}(E(e_1, e_2)) \cong \mathbb{N} \left[ \Sigma^{\pm 1}, \{ L(i, \delta_0, \delta_1) : i \in \mathbb{N}, \delta_0, \delta_1 \in \{0, 1\} \} \right] \text{ modulo relations},
\]
...where the relations are as follows:

\[
L(0, 0, 0) = 1,
L(i, 0, 0)L(j, 0, 0) = L(i + j, 0, 0),
L(i, 0, 0)L(j, 0, 1) = L(j, 0, 1),
L(i, 0, 0)L(j, 1, 0) = \sum^j L(j, 1, 0),
L(i, 0, 0)L(j, 1, 1) = L(j, 1, 1),
L(i, 0, 1)L(j, 1, 0) = 0,
L(i, 0, 1)L(j, 1, 1) = \sum^{j+1} L(i, 0, 1),
L(i, 1, 0)L(j, 1, 1) = \sum L(i, 1, 0),
L(i, 1, 1)L(j, 1, 1) = \sum L(i + j + 1, 1, 1),
L(i, 0, 1)L(j, 0, 1) = L(i, 0, 1)(1 + \sum^{j+1}) \text{ if } i \leq j,
L(i, 1, 0)L(j, 1, 0) = L(i, 1, 0)\sum(1 + \sum^{j-1}) \text{ if } i \leq j.
\]
Now given an $E(Sq^2)$-module $M$, you’d like to know how many $A(1)$-modules $N$ there are, such that $N \otimes_{A(1)} E(Sq^2) \cong M$. 
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In other words, you’d like to know the preimage of $[M] \in \text{Rep}(E(Sq^2))$ under the induction map $\text{Rep}(A(1)) \to \text{Rep}(E(Sq^2))$. 
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If the extension

$$1 \to E(1) \to A(1) \to E(Sq^2) \to 1$$

weren’t twisted—in other words, if $A(1)$ were isomorphic to $E(1) \otimes_{\mathbb{F}_2} E(Sq^2)$ as a Hopf algebra—then we could use classical deformation theory of modules.
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- Some important example cases.
- Classical (untwisted) deformation theory of modules.
- Hochschild (co)homology.
Let $k$ be a field, $A$ an associative unital $k$-algebra, $M$ an $A$-module. We have the extension of rings

$$E(x) \rightarrow A \otimes_k E(x) \rightarrow A,$$

and we’d like to know the set of $A \otimes_k E(x)$-modules $N$ such that $N/x$ is isomorphic to $M$ and such that the underlying $E(x)$-module of $N$ is isomorphic to $M \otimes_k E(x) \cong M \oplus M\{x\}$, up to isomorphism of $A \otimes_k E(x)$-modules.
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This set is in bijection with $HH^1(A, \text{End}_k(M))$. 
The bijection works like this: if \( A \xrightarrow{\xi} \text{End}_k(M \oplus M\{x\}) \) gives us the desired \( A \)-action on \( M \oplus M\{x\} \), then write \( \xi \) as

\[
\xi(a)(m_0 + m_1x) = \xi_0(a)(m_0 + m_1x) + \xi_1(a)(m_0 + m_1x)x.
\]

Unitality and associativity of the \( A \)-action determine \( \xi_0(a)(m_0) \), \( \xi_0(a)(m_1x) \), and \( \xi_1(a)(m_1x) \) uniquely; but \( \xi_1(a)(m_0) \) isn’t determined, and associativity of the \( A \)-action forces \( \xi_1 : A \rightarrow \text{End}_k(M) \) to be a Hochschild 1-cocycle.
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Hochschild (co)homology.
Here we’re using the standard Hochschild cochain complex. If $A$ is an associative unital $k$ algebra and $M$ is an $A$-bimodule, the Hochschild cochain complex is

$$\text{hom}_k(k, M) \xrightarrow{d^0} \text{hom}_k(A, M) \xrightarrow{d^1} \text{hom}_k(A \otimes_k A, M) \xrightarrow{d^2} \ldots$$

where

$$d^n f(a_0 \otimes \cdots \otimes a_n) = a_0 f(a_1 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n} (-1)^n f(a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n)$$

$$+ (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) a_n.$$
It’s also worth describing the standard Hochschild chain complex:

\[ M \xleftarrow{d_0} M \otimes_k A \xleftarrow{d_1} M \otimes_k A \otimes_k A \xleftarrow{d_2} \ldots \]

where

\[ d_n(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n \]

\[ + \sum_{i=1}^{n-1} (-1)^i m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \]
It’s not hard to see that the Hochschild chain complex is really the Dold-Kan construction applied to some simplicial abelian group. Namely, if $A$ is flat over $k$, then $HH_*(A, A) \cong \pi_*(S^1 \otimes HA)$, where the $S^1$ is any simplicial model for the circle, and the tensoring is using the usual tensoring of commutative $Hk$-algebras over simplicial sets.
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This is really concrete! (Time to draw pictures using chalk.)
Now if $X$ is any simplicial set, $\pi_*(X \otimes A)$ is the “higher-order Hochschild homology” defined by Pirashvili and studied by him and others. For example:

$$S^1 \otimes A \cong THH(A, A),$$

$$\operatorname{colim}_n \Omega^n(S^n \otimes A) \cong TAQ(A, A).$$
Now if $X$ is any simplicial set, $\pi_\ast(X \otimes A)$ is the "higher-order Hochschild homology" defined by Pirashvili and studied by him and others. For example:

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Does higher-order Hochschild (co)homology bear some relationship to deformations of modules?
Given an extension of Hopf algebras over a field $k$

$$1 \to E(n) \to A \to B \to 1,$$

where $E(n)$ is exterior on $n+1$ generators, and a $B$-module $M$, by a *twisted $n+1$-dimensional first-order deformation of $M$* we mean an $A$-module $N$ such that $N \otimes_A B = M$ and the underlying $E(n)$-module of $N$ is just $M \otimes_k E(n)$. 
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(The “twist” is by the Singer monodromy action of $B$ on $E(n)$.)
In the case of the extension

\[ 1 \to E(1) \to A(1) \to E(Sq^2) \to 1, \]

the monodromy action is \( Sq^2(Q_0) = Q_1 \) and \( Sq^2(Q_1) = 0. \)
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This “wants to be” classified by a cocycle in a cochain complex corresponding to \( K\text{III} \otimes E(Sq^2) \), with coefficients in \( \text{End}(M) \).
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And for good reason: they often cannot exist with coefficients in a nonsymmetric bimodule. (Example with $S^2$ drawn on chalkboard.)
Suppose $X$ is a simplicial pointed set, $k$ a field, $A$ an associative unital $k$-algebra, $M$ an $A$-bimodule with left action $\phi_\ell$ and right action $\phi_r$. For each integer $i \leq 1$ and each nonnegative integer $j \leq i$, let $\phi_{i,j} : M \otimes_k A \to M$ be a map of $k$-modules, and suppose that $\phi_{i,0} = \phi_r$ and $\phi_{i,i} = \phi_\ell^\text{op}$ for all $i$. Suppose further that each $\phi_{i,j}$ is unital, that is, $\phi_{i,j}(m \otimes 1) = m$ for any $m$. 
Let $X$ be a simplicial pointed finite set; we write $X_i^\times$ for the set of non-basepoint elements in the pointed set $X_i$. Choose a total ordering on $X_i^\times$ for each $i$. Let $(A \otimes X).$ be the functor $\mathcal{O}^{\text{op}} \to \text{Mod}(A)$ defined as follows:
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- $(A \otimes X.)_i$ consists of the tensor product $M \otimes_k A \otimes_k \cdots \otimes_k A$, with one tensor factor of $A$ for each non-basepoint element in $X_i$. We will write $m \otimes \{a_x\}_{x \in X_i^\times}$ for the element of $(A \otimes X.)_i$ whose component in the tensor factor $M$ is $m \in M$ and whose component in the tensor factor of $A$ corresponding to a non-basepoint element $x \in X_i^\times$ is $a_x$. 

The $j$th degeneracy map $s_j : (A \otimes X.)_i \to (A \otimes X.)_{i+1}$ sends $m \otimes \{a_x\}_{x \in X_i^\times}$ to the element $m \otimes \{b_y\}_{y \in X_{i+1}^\times}$, where

$$b_y = \begin{cases} 
  a_x & \text{if } s_j(x) = y \text{ in } X. \\
  1 & \text{if } y \neq s_j(x) \text{ for all } x \in X_i.
\end{cases}$$

This formula for the degeneracy is well-defined since at most one $i$-simplex has $j$th degeneracy a given $i + 1$-simplex, since the degeneracy maps of $X$ must admit the face maps as sections and hence the degeneracy maps are monomorphisms.
The $j$th face map $d_j : (A \otimes X.)_i \to (A \otimes X.)_{i-1}$ sends 
$m \otimes \{b_y\}_{y \in X_i^x} \text{ to } n \otimes \{a_x\}_{x \in X_{i-1}^x}$, where 

\[ n = \phi_{i,j}(m \otimes \text{(ordered product of all } b_y \text{ such that } d_j(y) \text{ is the basepoint in } X_{i-1})) \]

\[ a_x = \text{ordered product of all } b_y \text{ such that } d_j(y) = x. \]

Here the ordering is the ordering on the set of $i$-simplices $y \in X_i$. 
Now does this define a simplicial abelian group? And does its chain-homotopy equivalence class depend on the choice of ordering or on the $\phi_{i,j}$?
Now does this define a simplicial abelian group? And does its chain-homotopy equivalence class depend on the choice of ordering or on the $\phi_{i,j}$?

We must verify five simplicial identities for $(X \otimes A)$:

\begin{align*}
  d_k d_j &= d_{j-1} d_k \text{ if } k < j, \quad (1) \\
  d_k s_j &= s_{j-1} d_k \text{ if } k < j, \quad (2) \\
  d_j s_j &= \text{id} = d_{j+1} s_j, \quad (3) \\
  d_k s_j &= s_j d_{k-1} \text{ if } k > j + 1, \quad (4) \\
  s_k s_j &= s_{j+1} s_k \text{ if } k \leq j. \quad (5)
\end{align*}