Write your name here: ANSWER ICAY

Differential equations and linear algebra, quiz 2; fall term 2015.
Instructor: A. Salch

Some instructions:
- Show your work on all problems. You need to include at least enough detail to make it clear to the grader what your line of reasoning is, in arriving at your answer, for each problem on the exam. There is the possibility of receiving partial credit on problems where you have not arrived at the correct answer but made some progress toward the correct answer.
- You can write with a pencil or a pen of any color other than red. You may not use any of the following things: computers, calculators, phones, iPods or other digital devices, or any human being other than yourself. You can use any books or notes you have with you.
- You probably don’t need to be reminded of this: don’t cheat. If you are caught cheating, you will (for starters) receive no credit on this exam, and you will have to go through the university’s disciplinary process, including an expulsion hearing. This is a lot worse than failing this exam or even failing this entire class.

For graders’ use:

<table>
<thead>
<tr>
<th>Problem number</th>
<th>Points received</th>
<th>Points possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Total</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>
Problem 1. Convert the ODE
\[
\frac{dy}{dx} = \frac{-4y^6 - 5x}{6xy^5},
\]
to a differential 1-form, find an integrating factor for it, and solve the ODE for \( x > 0 \).

\[
(4y^6 + 5x) \, dx + 6xy^5 \, dy = 0
\]

\[
\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} = \frac{24y^5 - 6y^5}{6xy^5} = \frac{3}{x}, \text{ which depends only on } x.
\]

\[
\mu(x) = e^{\int \frac{3}{x} \, dx} = e^{\ln|x|^3} = x^3
\]

\[
(4x^3y^6 + 5x^4) \, dx + 6x^4y^5 \, dy = 0 \quad \text{is now exact.}
\]

\[
\int (4x^3y^6 + 5x^4) \, dx = x^4y^6 + x^5 + c_1(x)
\]

\[
\int 6x^4y^5 \, dy = x^4y^6 + c_2(x)
\]

\[
F(x,y) = |x^4y^6 + x^5 + c| = 0
\]

or:
\[
x^4y^6 = -x^5 - c
\]

\[
y = \pm \sqrt[6]{-x - \frac{c}{x^4}}
\]

either way is fine.
Problem 2. Use a substitution to solve the ODE
\[ \frac{dy}{dx} = \frac{x + 2y}{2x + y} \]
(As you did on the homework, it is okay to give an implicit answer, i.e., an algebraic equation relating \( y \) and \( x \) which you cannot solve explicitly for \( y \).)

\[ v = \frac{y}{x}, \quad y = vx \]

\[ \frac{dy}{dx} = \frac{d}{dx} (vx) = v + x \frac{dv}{dx} = \frac{x + 2y}{2x + y} = \frac{x + 2(vx)}{2x + vx} = \frac{1 + 2v}{2 + v} \]

\[ x \frac{dv}{dx} = \frac{1 + 2v}{2 + v} - v = \frac{1 + 2v - 2v - v^2}{2 + v} = \frac{1 - v^2}{2 + v} \]

\[ \int \frac{1}{x} \, dx = \int \frac{2 + v}{-v^2 + 1} \, dv = \ln |x| + C \]

\[ u = v^2 + 1 \]
\[ \frac{du}{dv} = 2v \]
\[ dv = \frac{1}{2v} \, du \]

\[ \frac{2}{(v+1)(v-1)} = \frac{A}{v+1} + \frac{B}{v-1} \]

\[ A + B = 0 \]
\[ R \cdot A = 2 \]

\[ 2 = (v-1) A + (v+1) B \]
\[ R = -A, \quad -2A = 2 \Rightarrow A = -1, \quad B = 1 \]
\[
\begin{align*}
\ln|x| + C &= \int \frac{2}{\sqrt{v^2 - 1}} \, dv + \int \frac{v}{\sqrt{v^2 - 1}} \, dv \\
&= \int \frac{-1}{v+1} + \frac{1}{v-1} \, dv + \int \frac{v}{2v} \, du \\
&= -\ln|v+1| + \ln|v-1| - \frac{1}{2} \ln|u| \\
&= -\ln\left|\frac{y}{x} + 1\right| + \ln\left|\frac{y}{x} - 1\right| - \frac{1}{2} \ln|\left(\frac{y}{x}\right)^2 - 1| \\
&= \ln\left|\frac{y}{x} - 1 - \sqrt{\left(\frac{y}{x}\right)^2 - 1}\right| \\
either \ \text{way} & \ \text{is fine!}
\end{align*}
\]
Problem 3. Solve the initial value problem
\[
\begin{align*}
\frac{dy}{dx} + y &= xy^3 \\
y(0) &= 1.
\end{align*}
\]
(Hint: try a substitution.)

\[v = y^{-2}, \quad \frac{dv}{dx} = \frac{d}{dx}(y^{-2}) = -2y^{-3}\frac{dy}{dx}\]

\[\frac{dy}{dx} = \frac{1}{2} y^3 \frac{dv}{dx} + y\]

\[
\frac{dv}{dx} - 2v^{-2} = -2x
\]

\[v = \frac{\int -2xe^{\int -2dx} \, dx}{e^{\int -2dx}} = \frac{-2\int x(e^{\int -2dx})^{-2} \, dx}{(e^{\int -2dx})^{-2}} \]

\[= -2\int xe^{-2x} \, dx \]

\[= \text{ven/} \]
\[ su'v' = uv' - su'v \]

\[ u = x \quad v = \frac{1}{2} e^{-2x} \]

\[ u' = 1 \quad v' = e^{-2x} \]

\[ \int xe^{-2x} \, dx = \frac{-x}{2} e^{-2x} + \frac{1}{2} \int e^{-2x} \, dx \]

\[ = \frac{-x}{2} e^{-2x} + \frac{1}{4} e^{-2x} + C \]

\[ = \frac{-1}{2} \left( x + \frac{1}{2} \right) e^{-2x} + C. \]

\[ y^{-2} = v = -2 \int xe^{-2x} \, dx \]

\[ = -2 \left( \frac{-1}{2} \left( x + \frac{1}{2} \right) e^{-2x} + C \right) \]

\[ = x + \frac{1}{2} - 2ce^{2x}. \]

\[ y(0) = 1 \quad 1 = 0 + \frac{1}{2} - 2ce^{0} = \frac{1}{2} - 2c \]

\[ c = \frac{1}{4} \]

\[ y = \sqrt{-x + \frac{1}{2} + \frac{1}{2} e^{2x}} \]