Write your name here:  

Answer_key

Differential equations and linear algebra, quiz 1, fall term 2015.  
Instructor: A. Salch

Some instructions:
- Show your work on all problems. You need to include at least enough detail to make it clear to the grader what your line of reasoning is, in arriving at your answer, for each problem on the exam. There is the possibility of receiving partial credit on problems where you have not arrived at the correct answer but made some progress toward the correct answer.
- You can write with a pencil or a pen of any color other than red. You may not use any of the following things: computers, calculators, phones, iPods or other digital devices, or any human being other than yourself. You can use any books or notes you have with you.
- You probably don’t need to be reminded of this: don’t cheat. If you are caught cheating, you will (for starters) receive no credit on this exam, and you will have to go through the university’s disciplinary process, including an expulsion hearing. This is a lot worse than failing this exam or even failing this entire class.

For graders’ use:

<table>
<thead>
<tr>
<th>Problem number</th>
<th>Points received</th>
<th>Points possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Total</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1. Find the general solution to the differential equation

\[ \frac{dy}{dx} + (\cos x) y = \cos x. \]

\[
y = \left( \int \cos x \ e^{\cos x \ dx} \ dx \right) / e^{\cos x \ dx} \\
= \left( \int \cos x \ e^{\sin x} \ dx \right) / e^{\sin x} \\
= \frac{e^{\sin x}}{e^{\sin x}} + \frac{e^{\sin x}}{e^{\sin x}} \\
= 1 + \frac{e^{\sin x}}{e^{\sin x}}.
\]
Problem 2. Solve the initial value problem:
\[
\frac{dy}{dx} = xy^2 \cos x
\]
\[y(0) = 1.\]

\[
\int y^2 \, dy = \int x \cos x \, dx
\]
\[
= -y^{-1} = x \sin x - \int \sin x \, dx
\]
\[
= x \sin x + \cos x + c.
\]

\[
y = \frac{-1}{x \sin x + \cos x + c}.
\]

\[y(0) = 1 = \frac{-1}{1 + c}.
\]

\[c = -2.
\]

\[
y = \frac{-1}{x \sin x + \cos x - 2}.
\]
Problem 3. A hot piece of metal, with temperature 200 degrees Celsius, is dropped into a tank of coolant fluid. The coolant fluid is kept at a constant temperature of 0 degrees Celsius. Suppose that, after one minute in the coolant fluid, the temperature of the piece of metal is 100 degrees Celsius. Use Newton’s law of cooling to set up a differential equation for the temperature $T(t)$ of the piece of metal after $t$ minutes, and solve the differential equation. Make sure you solve for all the constants that appear in your solution.

\[
\frac{dT}{dt} = (T - 0)k
\]

\[
\int \frac{1}{t} \, dt = \int k \, dt = kt + c
\]

\[
= \ln |T| = \ln 100
\]

\[
T = \pm e^{kt+c}
\]

\[
T(0) = 200 : \quad 200 = \pm e^c \implies c = \ln 200, \quad "t = +"
\]

\[
T(1) = 100 : \quad 100 = e^{k+\ln 200} = 200e^k \implies \frac{1}{2} = e^k
\]

\[
k = \ln \frac{1}{2}
\]

\[
T = e^{\left(\ln \frac{1}{2}\right)t + \ln 200} = (e^{\ln \frac{1}{2}})t \cdot e^{\ln 200} = 200 \left(\frac{1}{2}\right)t.
\]
**Problem 4.** Find the general solution to the ODE

\[
\frac{dx}{dt} + \frac{1}{t} x + \frac{1}{t(t+1)} = 0
\]

when \( t > 0 \).

\[
\frac{dx}{dt} + \frac{1}{t} x = \frac{-1}{(t+1)t^2}
\]

\[
x = \left( \int \frac{-1}{(t+1)t^2} e^{\int \frac{1}{t} \, dt} \, dt \right) / e^{\int \frac{1}{t} \, dt}
\]

\[
= \left( \int \frac{-1}{(t+1)t^2} \ln t \, dt \right) / e^{\ln t}
\]

\[
= \left( \int \frac{-1}{t(t+1)} \, dt \right) / t
\]

\[
\frac{-1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}
\]

\[-1 = A(t+1) + Bt = (A+B) t + A
\]

\[A = -1, \quad B = 1
\]

\[
\int \frac{-1}{t(t+1)} \, dt = \left( \int \frac{1}{t} \, dt + \int \frac{1}{t+1} \, dt \right) / t
\]

\[
= -\ln t + \ln \left( \frac{t+1}{t} \right) + c
\]

\[
= \ln \left( \frac{t+1}{t} \right) / t + c
\]

(either way is fine.)