1. A simple example from celestial mechanics.

For the first lecture, let’s think about a very classical problem in physics: suppose you have two gravitationally attracting bodies, and you want to describe how they move around under the force of gravity. For the first lecture, let’s make a lot of simplifying assumptions: suppose one of the two bodies is much more massive than the other, so that we can treat the larger one as fixed and (without introducing too much error) ignore the effect of the smaller’s one gravity on the larger one. Let’s call the more massive body “the star” and the less massive body “the planet.”

Let’s also assume, for now, that the two bodies are constrained to move in a single line. (All these simplifying assumptions we are making can be lifted, and later on in the course, we will be lifting them; but for now, it’s worth making these simplifying assumptions, as they make this first example much clearer.) So we have a star, which remains fixed, and a planet; let’s write \( p \) (standing for “position”) for the distance between the planet and the star.

It turns out you actually already know how to do this, although you might not realize you know it. Remember Newton’s second law,

\[ F = ma, \]

describing the relation between force \( F \) exerted on a body, the mass \( m \) of the body, and the acceleration \( a \) of the body; and Newton’s law of gravitation,

\[ F = \frac{GMm}{d^2}, \tag{1.0.1} \]

where \( F \) is the force of gravitational attraction between two massive bodies, \( G \) is the universal gravitational constant, \( M \) is the mass of one of the two bodies, \( m \) is the mass of the other body, and \( d \) is the distance between them. I feel quite confident that you have seen these formulas before! Now all you have to do is to put them together: let \( M \) be the mass of the star, and \( m \) the mass of the planet, \( F \) the gravitational force the star exerts on the planet, and remember that \( p \) denotes the distance between the two bodies, so we can plug in \( p \) for \( d \) in equation 1.0.1. Consequently we get the equation

\[ F = ma = \frac{-GMm}{p^2}, \]

with the minus sign to make sure gravity is working in the right direction (i.e., that it pulls the planet toward the star, instead of pushing it away), and dividing by \( m \) and then multiplying by \( p^2 \),

\[ ap^2 = -GM. \]
Now remember that acceleration is just the second derivative $\frac{d^2 p}{dt^2}$ of position with respect to time! So our equation now reads

$$p^2 \frac{d^2 p}{dt^2} = -GM,$$

or if you prefer to write $p''$ for $\frac{d^2 p}{dt^2}$,

$$p^2 \cdot p'' = -GM.$$

Now $G$ and $M$ are both constant, so their product $GM$ is also a constant. To keep it simple, let’s write $c = -GM$ for that constant. Now we’re down a really simple equation:

$$p^2 \cdot p'' = c.$$

Now here’s the big question: do you know of any functions $p(t)$ with the property that, when you multiply $p^2$ by the second derivative of $p$, you get a constant? There’s a sort of stupid example, the zero function $p(t) = 0$, but if you go looking for other examples of functions $p(t)$ such that $p^2 \cdot p''$ is constant, you will not find any by just guessing.

2. Solving the differential equation.

The equation $p^2 \cdot p'' = c$ is an equation involving the derivatives of a function $p$, namely, the second derivative $p''$ of $p$, and the zeroth derivative of $p$, which is just $p$ itself. We call such a thing a differential equation. To “solve” the equation $p^2 \cdot p'' = c$ means to find the functions $p(t)$ that actually satisfy the differential equation. It turns out that this particular differential equation $p^2 \cdot p'' = c$ is a little harder to solve than the ones we will spend the first 3/4 of the semester working with, so I’m going to tell you how to do it, but we won’t come back to this method for solving differential equations until nearly the end of the semester.

The idea is to think back on what you learned in your second semester of calculus: if $p(t)$ is a smooth function, meaning that all its derivatives $\frac{dp}{dt}, \frac{d^2 p}{dt^2}, \frac{d^3 p}{dt^3}, \ldots$ exist, then $p(t)$ has a Taylor series at every choice of real number $t$. For simplicity, let’s take $t = 0$. “Usually” (i.e., unless $p(t)$ has some very pathological properties), $p(t)$ will be equal to its Taylor series inside the radius of convergence of the Taylor series, i.e.,

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$$

for some real numbers $a_0, a_1, a_2, \ldots$, for all $t$ in some open interval $(-\epsilon, \epsilon)$. You also know Taylor’s formula $a_n = \frac{p^{(n)}(0)}{n!}$.

Let’s just try to see what these numbers $a_0, a_1, a_2, \ldots$ have to be, in order for $p^2 \cdot p''$ to be equal to some constant $c$. Recall, from second-semester calculus, that you take derivatives of power series using the same “power rule” that you use to take the derivative of a polynomial, i.e.,

$$p'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \ldots$$

$$p''(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 60a_6 t^4 + \ldots$$
Now you have to keep dredging up painful memories of your second semester of calculus, to remember that to multiply two power series, you do this:

\[ p(t) \cdot p''(t) = a_0 \left( 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 60a_6 t^4 + \ldots \right) \]

\[ = a_1 t \left( 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 60a_6 t^4 + \ldots \right) \]

\[ = a_2 t^2 \left( 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 60a_6 t^4 + \ldots \right) \]

\[ = a_3 t^3 \left( 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 60a_6 t^4 + \ldots \right) \]

\[ = \ldots \]

This is a big mess, with infinitely many terms in this sum, but notice that, for each integer \( n \), there are only finitely many monomial terms involving \( t^n \). So you “gather like terms” to get:

\[ p(t) \cdot p''(t) = 2a_0a_2 + (6a_0a_3 + 2a_1a_2)t + (12a_0a_4 + 6a_1a_3 + 2a_2^2)t^2 + (20a_0a_5 + 12a_1a_4 + 8a_2a_3)t^3 + \ldots. \]

We have to multiply by \( p \) one more time to get \( p^2 \cdot p'' \):

\[ (p(t))^2 \cdot p''(t) = a_0^2(2a_0a_2 + (6a_0a_3 + 2a_1a_2)t + (12a_0a_4 + 6a_1a_3 + 2a_2^2)t^2 + (20a_0a_5 + 12a_1a_4 + 8a_2a_3)t^3 + \ldots) \]

\[ + a_1(t(2a_0a_2 + (6a_0a_3 + 2a_1a_2)t + (12a_0a_4 + 6a_1a_3 + 2a_2^2)t^2 + (20a_0a_5 + 12a_1a_4 + 8a_2a_3)t^3 + \ldots) \]

\[ + a_2 t^2(2a_0a_2 + (6a_0a_3 + 2a_1a_2)t + (12a_0a_4 + 6a_1a_3 + 2a_2^2)t^2 + (20a_0a_5 + 12a_1a_4 + 8a_2a_3)t^3 + \ldots) \]

\[ + a_3 t^3(2a_0a_2 + (6a_0a_3 + 2a_1a_2)t + (12a_0a_4 + 6a_1a_3 + 2a_2^2)t^2 + (20a_0a_5 + 12a_1a_4 + 8a_2a_3)t^3 + \ldots) \]

\[ = 2a_0^2a_2 + (4a_0a_1a_2 + 6a_0^2a_3)t \]

\[ + (12a_0^2a_4 + 12a_0a_1a_3 + 4a_0a_2^2 + 2a_0^3a_2)t^2 \]

\[ + (20a_0^3a_5 + 24a_0a_1a_4 + 16a_0a_2a_3 + 6a_0^2a_3 + 4a_1a_2^2)t^3 \]

\[ = c. \]

Clearly you can compute the coefficient of \( t^4 \), the coefficient of \( t^5 \), and so on, for as long as you have the patience to do it, to get more terms in the Taylor series of the solutions \( p(t) \) to \( p \cdot p'' = c \). You can also, in this case and in many others, spot a pattern in the coefficients, and prove that it holds for all the coefficients, to describe all the coefficients. Anyway, now we get some equations:

\[ 2a_0^2a_2 = c, \]

\[ 4a_0a_1a_2 + 6a_0^2a_3 = 0, \]

\[ 12a_0^2a_4 + 12a_0a_1a_3 + 4a_0a_2^2 + 2a_0^3a_2 = 0, \]

\[ 20a_0^3a_5 + 24a_0a_1a_4 + 16a_0a_2a_3 + 6a_0^2a_3 + 4a_1a_2^2 = 0, \ldots \]

and now we solve them as far as we can (now is a good time to go back to writing \(-GM\) instead of \(c\), to return back to the physical interpretation for the pure mathematics we’re
doing):

\[ a_2 = \frac{-GM}{2a_0^2} \]
\[ a_3 = \frac{-4a_0a_1a_2}{6a_0^2} = \frac{2a_1GM}{3a_0^3} \]
\[ a_4 = \frac{-12a_0a_1a_3 - 4a_0a_2^2 - 2a_1^2a_2}{12a_0^5} = \frac{-a_0a_2^2GM - 2G^2M^2}{6a_0^5} \]

Notice what’s going on in these equations: we can’t solve for \( a_0 \) and \( a_1 \) or for \( G \) and \( M \), but we can solve for all the higher coefficients, \( a_n \) for \( n > 1 \), in terms of the numbers \( a_0, a_1, G \) and \( M \).

This makes perfect sense when you think about Taylor’s formula:

\[ a_0 = p(0), \]
\[ a_1 = p'(0), \]
\[ a_2 = \frac{p''(0)}{2}, \]
\[ a_3 = \frac{p'''(0)}{6}, \]
\[ a_4 = \frac{p''''(0)}{24}, \]
\[ \ldots \ldots \]

in other words, the Taylor coefficient \( a_0 \) is the initial distance \( p(0) \) between the planet and the star, and the Taylor coefficient \( a_1 \) is the initial velocity \( p'(0) = v(0) \) of the planet (holding the star fixed). The initial distance and the initial velocity aren’t determined by just Newton’s second law and Newton’s law of gravitation; those can be almost any numbers (you just can’t let \( a_0 = 0 \)). But once you specify the initial distance \( a_0 \) and the initial velocity \( a_1 \) and the mass of the star \( M \) and the universal gravitational constant \( G \), everything else about the motion of the planet is determined by Newton’s second law and Newton’s law of gravitation: the position of the planet at all times \( t \) is equal to

\[ p(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \ldots \]
\[ = a_0 + a_1t + \frac{-GM}{2a_0^2}t^2 + \frac{2a_1GM}{3a_0^3}t^3 + \frac{-a_0a_2^2GM - 2G^2M^2}{6a_0^5}t^4 + \ldots \]

3. The general relationship between differential equations and problems in physics and engineering.

In this example we just did, we saw that, using Newton’s second law \( F = ma \) together with a description of the force \( F \) acting on the planet, we were able to extract a differential equation with the following properties:

- The position of the planet is a solution to the differential equation.
• The differential equation has one solution for each choice of initial conditions (initial position, initial velocity).

This works much more generally: given a physical system with a finite number of interacting bodies, using Newton’s second law \( F = ma \) together with a description (Newton’s law of gravitation, if you are working with gravity; or Maxwell’s equations, if you are working with electromagnetic force; etc.) of the forces acting on the bodies, you can reduce the situation to a system of differential equations (maybe not a single equation, but at least a finite set of equations), with the property that:

• Each solution to the system of differential equations describes the position of each of the objects.
• The system of differential equations has one solution for each choice of initial conditions.

We will be doing plenty more of this as the semester goes on, but this at least is the motivation for why we’re studying differential equations!