Natural Superconvergent Points of Equilateral Triangular Finite Elements – A Numerical Example

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Abstract. A numerical test case demonstrates that Lobatto and Gauss points are not natural superconvergent points for cubic and quartic finite elements under equilateral triangular mesh.

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1. Introduction.

Natural superconvergent points are those points where the rate of convergence exceeds the best possible global rate without post-processing. Research in this area started in the early 70s, and may even trace back to the late 60s (see [5, 6, 7, 8, 9, 12, 14] and references therein). Consider the C^0 finite element approximation, it is well known that the Lobatto points are superconvergent points for function values, and the Gauss points are for derivatives. This result is valid for the one dimensional case, as well as for the tenser-product space in higher dimensional settings. As for triangular elements, the situation is much more complicated. Earlier researches focused only on lower-order elements, namely, linear and quadratic elements under strongly regular meshes.

In the mid-90s, two systematic way to find superconvergent points were developed. One was the symmetry theory due to Schatz-Sloan-Wahlbin [11, 12]. By the symmetry theory, superconvergence occurs at mesh symmetry centers. One advantage of this theory is its generality, which is applicable to any dimensional situations. Nevertheless, it does not say that there is no other superconvergence points, and therefore it is not conclusive. Almost at the same time, another approach was proposed by Babuška-Strouboulis et al. They established a theoretical framework which narrows the task of locating superconvergence points to finding intersections of some polynomial contours on a "master cell". The actual procedure was carried out by a computer algorithm without explicitly constructing those polynomials, and 10 digits were provided in their reported data [2, 3]. This approach is called "compute-based proof". With help of computer, they predicted all derivative superconvergence points for the Poisson and the Laplace equations under four different triangular

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mesh patterns for polynomial finite element spaces of degree up to 7. The advantage of this approach is that it is conclusive.

Along the line of computer based proof, Zhang proposed an analytic approach which constructs explicitly the needed polynomials through an orthogonal decomposition [13]. Using this approach, Lin-Zhang studied superconvergent points for triangular elements [10]. Their results verify that the computed data for triangular elements in [3] have 9 digits of accuracy except one pair (with 8-7 accurate digits). In addition, they reported for the first time, superconvergent points for function values of the Laplace equation. The advantage of this approach is the following:

1. it offers quality of the superconvergent points. With polynomials explicitly given, one can easily verify how accurate those points are.

2. It can be readily verified by a reader. With a pencil and a piece of paper, an interested reader may check if those polynomials are indeed the periodic finite element solutions on the master cells. As for the superconvergent points, when polynomials are explicitly given, a root finding can be done analytically for lower-order case (which are the most interesting cases anyway).

3. It provides some more insights. A orthogonal polynomial basis functions on triangular meshes is constructed in a systematic way, which reveals the structure of the periodic finite element solution.

4. It can be generalized to 3D cases, where superconvergence results are relatively scarce.

As a related work, Chen discussed superconvergent points of triangular element by orthogonal decomposition and element analysis technique [5]. His result is consistent with symmetry theory and computer based proof.

According to the above review, investigation of natural superconvergent points for triangular elements seems finished, except for one special case, the equilateral triangulation. Yet, this is the most important case, since automatic mesh generators based on the Delaunay triangulation produce almost equilateral triangles for most portion of the mesh. There was one very interesting result due to Blum-Lin-Rannacher in 1986 [4] which revealed that convergent rate for $u - u_h$ at the element edge point for equilateral mesh is of h^4, which is two-orders higher that for the regular mesh pattern. Therefore, it is natural to ask if there is any other special properties with the equilateral triangular mesh for higher-order finite elements. Especially, since for quadratic element the Lobbato (Gauss) points are supercovergent points of function (derivative) values along element edges (this is true for both regular and equilateral triangulation [1]), one may wonder if it also true for cubic and quartic elements. The current paper provides a negative answer for this question. By a non-trivial numerical example, we demonstrate that Lobbato or Gauss points are not natural superconvergent points for cubic and quartic elements under the equilateral mesh. The convergence behavior is the same for regular and equilateral triangulations with only one exception, linear element.

2. A Numerical Example.

Define a equilateral triangular domain Ω enclosed by three straight lines $y = 0$, $y = \sqrt{3}x$, and $y = \sqrt{3}(1 - x)$. Consider the boundary value problem

$$-\Delta u = f, \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

We choose f such that $u = y(y - \sqrt{3}x)(y + \sqrt{3}x - \sqrt{3})e^{x+y}$, and then solve (2.1) by linear, quadratic, cubic, and quartic finite elements under uniform triangulation of Ω by dividing each side of Ω to n sub-intervals. We choose $h^{-1} = n = 8$ (see Figure 1) for the initial mesh, and use the regular refinement with bi-section strategy.

We calculate the error $e_h = u - u_h$ at the Lobatto points and $\partial_t(u - u_h) = \vec{t} \cdot \nabla(u - u_h)$ at the Gauss points. Here $\vec{t}$ is the tangential unit vector. Table 1 lists Lobatto and Gauss points on $[-1, 1]$, and a suitbale liner mapping is used to compute the Lobatto and Gauss points on element edges. For each given $h = 1/n$, we denote $L(h)$, the set of all Lobatto points on element edges; and $G(h)$, the set of all Gauss points on element edges. Define the subdomain

$$\Omega_H = (x, y) \in \Omega : dist((x, y), \partial\Omega \geq H)$$

where $H > 0$ is a fixed constant (see Figure 2), and $H = 1/8$ in this particular example. Let ε_h denote the set of element edges in the mesh, and let us define the following measures

$$E_h(e_h, \Omega_H) = \max_{\boldsymbol{z} \in L(h) \cap \Omega_H} |(e_h)(\boldsymbol{z})|, \quad E_h(\partial_t e_h, \Omega_H) = \max_{\boldsymbol{z} \in G(h) \cap \Omega_H} |\partial_t(e_h)(\boldsymbol{z})|;$$

$$\bar{E}_h(e_h, \Omega_H) = \frac{1}{|L(h)|} \sum_{\boldsymbol{z} \in L(h) \cap \Omega_H} |(e_h)(\boldsymbol{z})|, \quad \bar{E}_h(\partial_t e_h, \Omega_H) = \frac{1}{|G(h)|} \sum_{\boldsymbol{z} \in G(h) \cap \Omega_H} |\partial_t(e_h)(\boldsymbol{z})|;$$

$$\tilde{E}_h(e_h, \Omega_H) = \sqrt{\sum_{\ell \in \varepsilon_h \cap \Omega_H} \int_\ell e_h^2 ds}, \quad \tilde{E}_h(\partial_t e_h, \Omega_H) = \sqrt{\sum_{\ell \in \varepsilon_h \cap \Omega_H} \int_\ell (\partial_t e_h)_h^2 ds}.$$

Here $|G(h)|$ is the cardinal value, i.e., the number of points in $G(h)$. The integrals in $\tilde{E}_h(e_h, \Omega_H)$ and $\tilde{E}_h(\partial_t e_h, \Omega_H)$ are computed using Gauss-Lobatto quadrature and Gauss-Legendre quadrature, respectively. Table 2 and Table 3 collect all computed error values for different $L(h)$ and $G(h)$, respectively, where the relative graphs (in log-log scale) are plotted in Figures 3-6. All measures of e_h or $\partial_t e_h$ lead to the same conclusion, and hence we focus on $E_h(e_h, \Omega_H)$ and $E_h(\partial_t e_h, \Omega_H)$. Based on the computed data, we summarize our results in Table 4 and draw following conclusion:

1. For cubic and quartic finite element approximation of the Poisson equation, the Lobatto points are not superconvergent points for $u - u_h$, and the Gauss points are not superconvergent points for $\partial_t(u - u_h)$.

2. For, quadratic, cubic, and quartic elements, the convergence behavior of the equilateral mesh is the same a the regular triangular mesh.

3. The only special case for the equilateral mesh is for linear element when the convergent rate is h^4 for nodal values.

4. Convergence behavior for average error....

Remark 3 1. Mesh symmetry points in Ω_H are element vertices and mid-edges.

2. If we replace Ω_H with Ω in all the previous error measures, the superconvergence is lost in two cases: superconvergence for e_h at mesh symmetry points in quartic element and superconvergence for $\partial_t e_h$ at mesh symmetry points in quartic element.

Element order	Labatto points	Gaussian points
1	± 1	0
2	$0, \pm 1$	$\pm \frac{1}{\sqrt{3}}$
3	$\pm \frac{1}{\sqrt{5}}, \pm 1$	$0, \pm\sqrt{\frac{3}{5}}$
4	$0, \pm\sqrt{\frac{3}{7}}, \pm 1$	$\pm\sqrt{\frac{3}{7} \pm \frac{4}{7}\sqrt{\frac{3}{10}}}$

Table 1: Lobatto and Gauss points on $[-1, 1]$ for elements of order 1 through 4

n	Cubic element			Quartic element		
	$\tilde{E}_h(e_h, \Omega_H)$	$\bar{E}_h(e_h, \Omega_H)$	$E_h(e_h, \Omega_H)$	$\tilde{E}_h(e_h, \Omega_H)$	$\bar{E}_h(e_h, \Omega_H)$	$E_h(e_h, \Omega_H)$
8	9.1561E-007	1.3541E-006	4.3632E-006	2.9158E-008	2.0899E-008	5.6626E-008
16	1.2569E-007	7.2703E-008	3.2206E-007	2.0769E-009	6.6178E-010	2.2385E-009
32	1.2833E-008	4.3698E-009	2.1923E-008	1.1056E-010	2.0852E-011	7.8289E-011
64	1.1545E-009	2.6768E-010	1.3691E-009	4.8401E-012	6.5100E-013	2.4815E-012

Table 2: Various error measures for e_h in Ω_h

n	Cubic element			Quartic element		
	$\tilde{E}_h(\partial_t e_h, \Omega_H)$	$\bar{E}_h(\partial_t e_h, \Omega_H)$	$E_h(\partial_t e_h, \Omega_H)$	$\tilde{E}_h(\partial_t e_h, \Omega_H)$	$\bar{E}_h(\partial_t e_h, \Omega_H)$	$E_h(\partial_t e_h, \Omega_H)$
8	1.0191E-004	8.6471E-005	1.6244E-004	1.5243E-006	1.3290E-006	3.3746E-006
16	2.8295E-005	1.0704E-005	2.4089E-005	2.1569E-007	8.6482E-008	2.5139E-007
32	5.9710E-006	1.3384E-006	3.2692E-006	2.2936E-008	5.5223E-009	1.7008E-008
64	1.0371E-006	1.6634E-007	4.0779E-007	2.0076E-009	3.4686E-010	1.0613E-009

Table 3: Various error measures for $\partial_t e_h$ in Ω_h

Element order	**Convergence rate for $E_h(e_h, \Omega_H)$**		**Convergence rate for $E_h(\partial_t e_h, \Omega_H)$**	
	Lobatto Points	**Symmetry points**	**Gaussian Points**	**Symmetry points**
1	4 (superconvergence)	2 (superconvergence)	2 (no superconvergence)	2 (no superconvergence)
2	4 (superconvergence)	4 (superconvergence)	3 (superconvergence)	2 (no superconvergence)
3	4 (no superconvergence)	4 (no superconvergence)	3 (no superconvergence)	4 (superconvergence)
4	5 (no superconvergence)	6 (superconvergence)	4 (no superconvergence)	4 (no superconvergence)

Table 4: Convergence rates in Ω_h

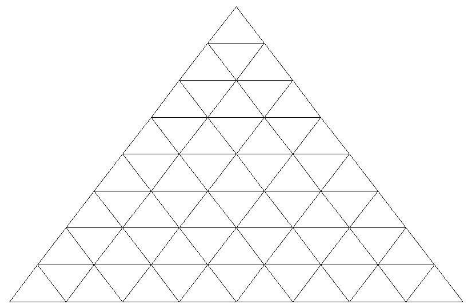

Figure 1: Initial mesh when $n = 8$

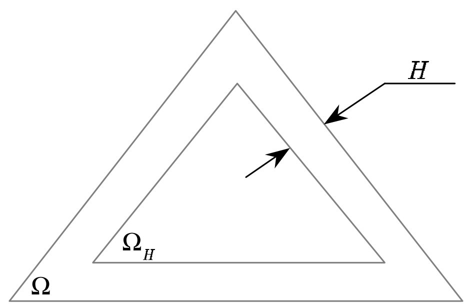

Figure 2: The subdomain Ω_H

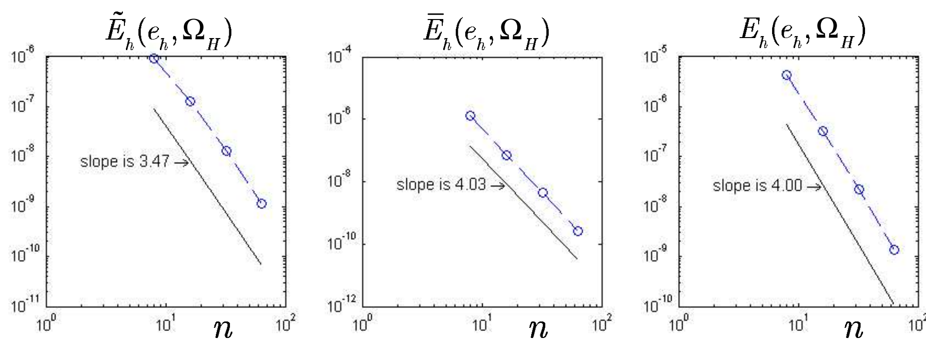

Figure 3: Various error measures for e_h in Ω_H - cubic element

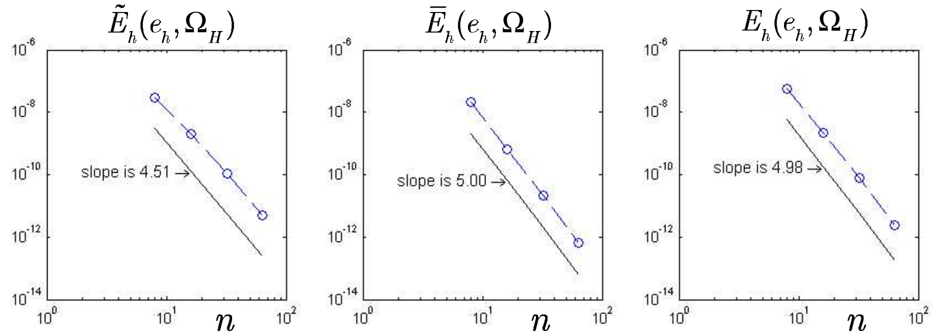

Figure 4: Various error measures for e_h in Ω_H - quartic element

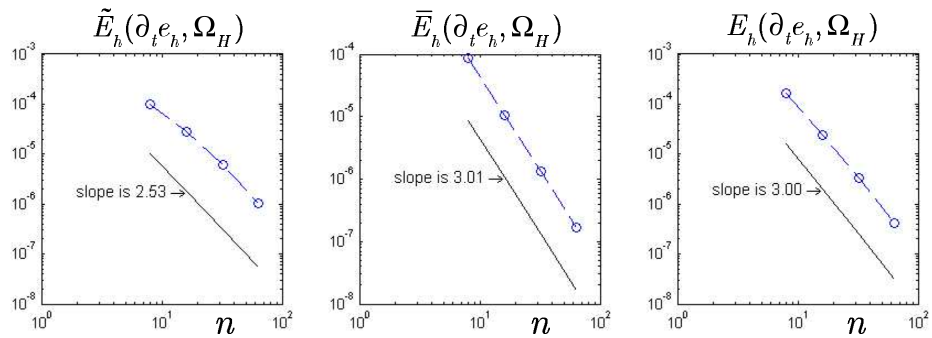

Figure 5: Various error measures for $\partial_t e_h$ in Ω_H - cubic element

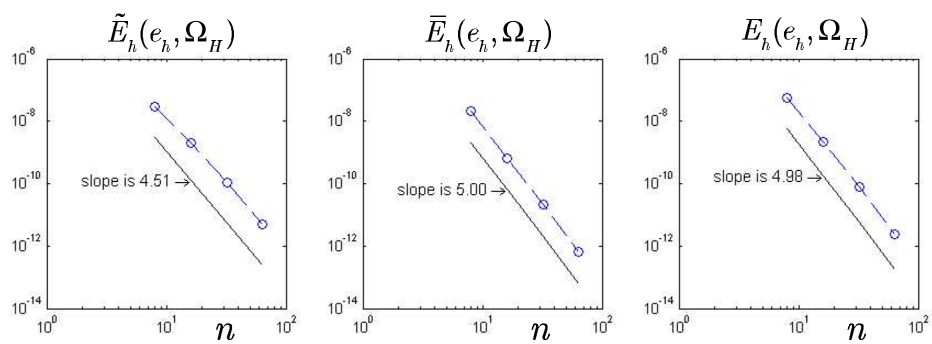

Figure 6: Various error measures for $\partial_t e_h$ in Ω_H - quartic element

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