DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **9**, Number **1**, January **2008**

Website: http://aimSciences.org

pp. **163–182**

THE PATCH RECOVERY FOR FINITE ELEMENT APPROXIMATION OF ELASTICITY PROBLEMS UNDER QUADRILATERAL MESHES

Zhong-Ci Shi and Xuejun Xu

LSEC, Institute of Computational Mathematics
Chinese Academy of Sciences
P.O. Box 2719, Beijing 100080, China

Zhimin Zhang

College of Mathematics and Computer Science
Hunan Normal University, Changsha, Hunan 410081, China
Department of of Mathematics
Wayne State University, Detroit, MI 48202, USA

(Communicated by Yanping Lin)

Abstract. In this paper, some patch recovery methods are proposed and analyzed for finite element approximation of elasticity problems using quadrilateral meshes. Under a mild mesh condition, superconvergence results are established for the recovered stress tensors. Consequently, *a posteriori* error estimators based on the recovered stress tensors are asymptotically exact.

1. **Introduction.** In recent years, the adaptive method based on *a posteriori* error control has become one of the most popular numerical methods for solving PDEs. In this paper, we will focus our attention on a posteriori error estimators based on gradient recovery techniques. The most well-known gradient recovery method is the ZZ patch recovery method developed by Zienkiewicz-Zhu [23]. Recently, Zhang-Naga proposed a new recovery technique [17], which is different from the ZZ patch recovery. The main advantages of the new recovery technique are as follows: It is polynomial preserving under arbitrary meshes, a property not shared by the ZZ; it is superconvergent under minor mesh restrictions. In this paper, we will use the recovery techniques developed by ZZ and Zhang-Naga to construct *a posteriori* error estimators for linear elasticity problems.

While residual type estimators have been well researched even for elasticity problems, (cf. [7],[8]), we have not seen theoretical analysis on recovery type error estimates for these problems. Nevertheless, recovery type estimators are widely used

2000 *Mathematics Subject Classification.* Primary: 65N30. 65N15; Secondary: 65N12, 65D10, 41A10.

Key words and phrases. Finite element, Superconvergence, Gradient recovery, Linear elasticity.

The first two authors are supported in part by the special funds for major state basic research projects (973) under grant 2005CB321701 and the National Natural Sciences Foundation of China under grant 10731060.

The third author is supported in part by the US National Science Foundation under grants DMS-0311807 and DMS-0612908, and by the National Natural Sciences Foundation of China under grant 10571053.

in engineering application. It is confirmed numerically that these type error estimators are very effective for computation of elasticity problems [21, 23, 24]. In this paper, we will present some recovered stress tensors and show that the relevant *a posteriori* error estimators are asymptotically exact. The theoretical results given in this paper are numerically supported by ZZ in [21, 23, 24]. On the other hand, the analysis here generalizes the results in [20] for the second order elliptic problem. Analysis of other recovery techniques including the ZZ patch recovery for the second order elliptic equation with triangular meshes can be found in [15].

This paper is organized as follows: Section 2 describes the model problem under consideration. Section 3 introduces the finite element method followed by the derivation of the superconvergence error estimate in Section 4. Section 5 is devoted to construct the recovered stress tensors, and derive their corresponding superconvergence errors. Finally, in the last section, we will give the reliable *a posteriori* error estimators based on the recovered stress tensors.

2. **Elasticity problems.** Let $\Omega \subset R^2$ be a bounded convex polygonal domain with vertices S_i, edges Γ_{ij} connecting S_i and S_j. Here indices on Γ_{ij}, S_i are understood as integers modulo n. In this paper, we consider quadrilateral bilinear finite element approximation of pure displacement and pure traction elasticity problems.

For any $\mathbf{v} = (v_1, v_2) \in V \hat{=} (H^1(\Omega))^2$, we set

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}), \quad 1 \le i, j \le 2,$$
$$\sigma_{ij}(\mathbf{v}) = \lambda(\mathrm{div}\mathbf{v})\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{v}), \quad 1 \le i, j \le 2,$$

where the constants $\lambda \ge 0$ and $\mu > 0$ are the Lame coefficients, and $\delta_{ij} = 0 (i \ne j)$, $\delta_{ii} = 1$. Moreover, we set for any $\mathbf{v} \in V$, the expressions $(|v_1|_m^2 + |v_2|_m^2)^{\frac{1}{2}}$ and $(\|v_1\|_m^2 + \|v_2\|_m^2)^{\frac{1}{2}}$ for $(m = 0, 1)$ will still be denoted by $|\mathbf{v}|_m$ and $\|\mathbf{v}\|_m$, the expressions $\|\cdot\|_m$ and $|\cdot|_m$ represent respectively a norm and a seminorm over the Sobolev space $H^m(\Omega)$ (cf. [9] for details).

We define the bilinear form $a(\cdot,\cdot)$ on $V \times V$ by

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \int_\Omega \sum_{i,j=1}^{2} \sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) dx \\ &= \lambda \int_\Omega \mathrm{div}\mathbf{u}\,\mathrm{div}\mathbf{v} dx + 2\mu \int_\Omega \sum_{i,j=1}^{2} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) dx \\ &:= \lambda(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v}) + 2\mu(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})), \end{aligned}$$

and the linear form

$$(\mathbf{f}, \mathbf{v}) = \int_\Omega \sum_{i=1}^{2} f_i v_i dx \quad f_i \in L^2(\Omega).$$

For any $k \ge 1$, define the Sobolev space

$$\hat{\mathbf{H}}^k(\Omega) = \{\mathbf{v} \in H^k(\Omega)^2 : \int_\Omega \mathbf{v} dx = 0, \ \int_\Omega \mathrm{rot}\mathbf{v} dx = 0\},$$

where $\mathrm{rot}\mathbf{v} = \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}$. Then the weak formulation of the pure traction elasticity problem is to find unknown displacement $\mathbf{u} \in \hat{\mathbf{H}}^1(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \langle \mathbf{g}, \mathbf{v} \rangle \ \forall \mathbf{v} \in \hat{\mathbf{H}}^1(\Omega), \tag{2.1}$$

where the inner product $\langle \cdot, \cdot \rangle$ defines the boundary integration

$$\langle \mathbf{g}, \mathbf{v} \rangle = \sum_{i=1}^{n} \int_{\Gamma_i} \mathbf{g}_i \cdot \mathbf{v} ds.$$

For the above pure traction problem to be uniquely solvable, the following compatibility condition must be satisfied

$$(\mathbf{f}, \mathbf{v}) + \langle \mathbf{g}, \mathbf{v} \rangle = 0 \ \mathbf{v} \in \mathbf{RM},$$

where the space of infinitesimal rigid motions $\mathbf{RM}$ is defined as follows:

$$\mathbf{RM} = \{\mathbf{v} : \mathbf{v}^T = (a + bx_2, c - bx_1), \ a, b, c \in R\}.$$

For the pure displacement problem, the boundary value problem can be written as:

$$\begin{cases} -\mu \triangle \mathbf{u} + (\lambda + \mu)\mathrm{grad}(\mathrm{div}\mathbf{u}) & = \mathbf{f} \text{ in } \Omega \\ \mathbf{u} & = \mathbf{0} \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

It has the following weak formulation: Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$\mu(\mathrm{grad}\mathbf{u}, \mathrm{grad}\mathbf{v}) + (\mu + \lambda)(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}) \ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega). \tag{2.3}$$

By the Poincare inequality, we know that there exists a unique solution to the above pure displacement problem.

3. **Finite element discretization.** Let T_h be a partition of the domain Ω by convex quadrilaterals with the mesh size $h := \max_{K \in T_h} h_K$, h_K the longest edge of K. Let $\hat{K} = [-1, 1] \times [-1, 1]$ be the reference square with vertices $\hat{Z}_i$, and let K be a convex quadrilateral with vertices $Z_i^K(x_i^K, y_i^K)$, $i = 1, 2, 3, 4$. It is known that there exists a unique bilinear mapping F_K such that $F_K(\hat{K}) = K$, $F_K(\hat{Z}_i) = Z_i^K$ given by

$$x_1 = \sum_{i=1}^{4} x_i^K N_i, \ x_2 = \sum_{i=1}^{4} y_i^K N_i,$$

where

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta), \ N_2 = \frac{1}{4}(1 + \xi)(1 - \eta),$$
$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta), \ N_4 = \frac{1}{4}(1 - \xi)(1 + \eta).$$

By a simple manipulation, we can explicitly express the mapping F_K as:

$$x_1 = a_0 + a_1\xi + a_2\eta + a_3\xi\eta, \ x_2 = b_0 + b_1\xi + b_2\eta + b_3\xi\eta,$$

where by suppressing the index "K",

$$a_0 = (x_1 + x_2 + x_3 + x_4)/4, \ b_0 = (y_1 + y_2 + y_3 + y_4)/4;$$
$$a_1 = (-x_1 + x_2 + x_3 - x_4)/4, \ b_1 = (-y_1 + y_2 + y_3 - y_4)/4;$$
$$a_2 = (-x_1 - x_2 + x_3 + x_4)/4, \ b_2 = (-y_1 - y_2 + y_3 + y_4)/4;$$
$$a_3 = (x_1 - x_2 + x_3 - x_4)/4, \ b_3 = (y_1 - y_2 + y_3 - y_4)/4.$$

The Jacobi matrix of the mapping F_K is

$$(DF_K)(\xi,\eta) = \begin{bmatrix} \frac{\partial x_1}{\partial \xi} & \frac{\partial x_2}{\partial \xi} \\ \frac{\partial x_1}{\partial \eta} & \frac{\partial x_2}{\partial \eta} \end{bmatrix} = \begin{bmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{bmatrix}$$

The determinant of the Jacobi matrix is

$$J_K = J_K(\xi,\eta) = J_0^K + J_1^K\xi + J_2^K\eta, \tag{3.1}$$

where

$$J_0^K = a_1b_2 - b_1a_2,\ J_1^K = a_1b_3 - b_1a_3,\ J_2^K = b_2a_3 - a_2b_3.$$

The inverse of the Jacobi matrix is

$$\begin{bmatrix} \frac{\partial \xi}{\partial x_1} & \frac{\partial \eta}{\partial x_1} \\ \frac{\partial \xi}{\partial x_2} & \frac{\partial \eta}{\partial x_2} \end{bmatrix} = (DF_K)^{-1} = \frac{1}{J_K}\begin{bmatrix} b_2 + b_3\xi & -b_1 - b_3\eta \\ -a_2 - a_3\xi & a_1 + a_3\eta \end{bmatrix} := \frac{1}{J_K}X. \tag{3.2}$$

It can be shown that

$$J_0^K := J_K(0,0) = \frac{|K|}{4},\ J_K(\xi,\eta) > 0. \tag{3.3}$$

For any function $v(x_1,x_2)$ defined on K, we associate $\hat{v}(\xi,\eta)$ by

$$\hat{v}(\xi,\eta) = v(x_1(\xi,\eta), x_2(\xi,\eta)),\ \text{or}\ \hat{v} = v \circ F_K.$$

It is straightforward to verify that

$$\begin{aligned} \frac{\partial v}{\partial x_1} &= \frac{\partial \hat{v}}{\partial \xi}\frac{\partial \xi}{\partial x_1} + \frac{\partial \hat{v}}{\partial \eta}\frac{\partial \eta}{\partial x_1} \\ &= \frac{1}{J_K}[b_2 + b_3\xi, -b_1 - b_3\eta][\frac{\partial \hat{v}}{\partial \xi}, \frac{\partial \hat{v}}{\partial \eta}]^T \\ &= \frac{1}{J_K}B \cdot \hat{\nabla}\hat{v}, \end{aligned}$$

here $B := [b_2 + b_3\xi, -b_1 - b_3\eta]$. Similarly,

$$\begin{aligned} \frac{\partial v}{\partial x_2} &= \frac{\partial \hat{v}}{\partial \xi}\frac{\partial \xi}{\partial x_2} + \frac{\partial \hat{v}}{\partial \eta}\frac{\partial \eta}{\partial x_2} \\ &= \frac{1}{J_K}[-a_2 - a_3\xi, a_1 + a_3\eta][\frac{\partial \hat{v}}{\partial \xi}, \frac{\partial \hat{v}}{\partial \eta}]^T \\ &= \frac{1}{J_K}A \cdot \hat{\nabla}\hat{v}, \end{aligned}$$

here $A = [-a_2 - a_3\xi, a_1 + a_3\eta]$.

We denote the midpoints of Z_2Z_4 and Z_1Z_3 as O_1 and O_2, respectively and let P_i is the midpoint of edge $Z_iZ_{i+1},\ i = 1,2,3,4$, (Figure 1). We have

$$\begin{aligned} P_2P_4 &= \frac{1}{2}(x_2 + x_3 - x_4 - x_1, y_2 + y_3 - y_4 - y_1) = 2(a_1, b_1), \\ P_3P_1 &= \frac{1}{2}(x_3 + x_4 - x_1 - x_2, y_3 + y_4 - y_1 - y_2) = 2(a_2, b_2), \\ O_1O_2 &= \frac{1}{2}(x_1 + x_3 - x_2 - x_4, y_1 + y_3 - y_2 - y_4) = 2(a_3, b_3). \end{aligned}$$

Then

$$|P_2P_4| = 2(a_1^2 + b_1^2)^{\frac{1}{2}},\ |P_1P_3| = 2(a_2^2 + b_2^2)^{\frac{1}{2}},\ |O_1O_2| = 2(a_3^2 + b_3^2)^{\frac{1}{2}},$$

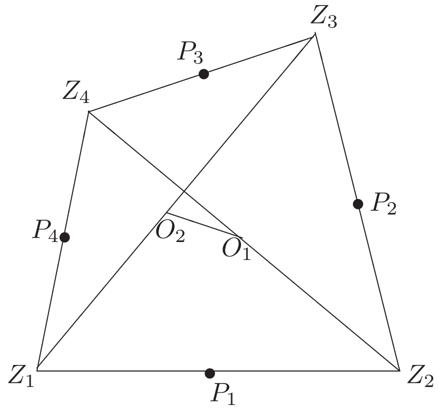

FIGURE 1. Quadrilateral

Moreover,it is easy to prove that

$$J_0^K = |\frac{1}{4}P_2P_4 \times P_1P_3| = \frac{1}{4}|P_2P_4||P_1P_3|\sin\alpha_K, \tag{3.4}$$

$$J_1^K = |\frac{1}{4}P_2P_4 \times O_1O_2| = \frac{1}{4}|P_2P_4||O_1O_2|\sin\beta_K, \tag{3.5}$$

$$J_2^K = |\frac{1}{4}P_1P_3 \times O_1O_2| = \frac{1}{4}|P_1P_3||O_1O_2|\sin\gamma_K, \tag{3.6}$$

where α_K is the acute angle between two lines P_1P_3 and P_2P_4, β_K is the acute angle between two lines P_2P_4 and O_1O_2, and γ_K is the acute angle between two lines P_1P_3 and O_1O_2.

Definition 3.1. A convex quadrilateral K is said to satisfy the **diagonal condition** if

$$d_K = |O_1O_2| = O(h_K^{1+\alpha}),\ \alpha > 0.$$

Note that K is a parallelogram if and only if $d_K = 0$.

Remark 1. Using d_K to characterize asymptotic quality of quadrilaterals in analysis can be traced back at least 20 years, cf. [14], also see [11] for more discussions on mesh conditions. Note that all quadrilaterals produced by a bi-section scheme of mesh subdivisions have the property $\alpha = 1$.

Definition 3.2. A partition T_h is said to satisfy **condition** α if there exist $\alpha > 0$ such that 1) any $K \in T_h$ satisfies the **diagonal condition**; and 2) any two $K_1,\ K_2$ in T_h that share a common edge satisfy a **neighboring condition**: For $j = 1, 2$

$$a_j^{K_1} = a_j^{K_2}(1 + O(h_{K_1}^\alpha + h_{K_2}^\alpha)),\ b_j^{K_1} = b_j^{K_2}(1 + O(h_{K_1}^\alpha + h_{K_2}^\alpha)). \tag{3.7}$$

Remark 2. Let us comment on the geometric meaning of **condition** α. Observe that

$$a_j^{K_1} = a_j^{K_2}, \quad b_j^{K_1} = b_j^{K_2}, \quad j = 1, 2$$

implies

$$P_2^{K_1}P_4^{K_1} = 2(a_1, b_1) = P_2^{K_2}P_4^{K_2}, \quad P_3^{K_1}P_1^{K_1} = 2(a_2, b_2) = P_3^{K_2}P_1^{K_2}.$$

Therefore, the neighboring condition characterizes how "similar" the neighbor quadrilaterals are. Together with the diagonal condition, quadrilaterals under **condition** α cannot deviate from parallelograms too much asymptotically. It is well known from the literature that parallelogram mesh means superconvergence.

Remark 3. We may use another parameter, say β, in **Definition 3.2** for the neighboring condition. Then we need to carry both parameters all the way in later statements and proofs, and have $\gamma = \min(\alpha, \beta)$ for the convergence rate. In order to simplify the matter, we understand from the very beginning that the α is the smaller power in the diagonal condition and the neighboring condition.

To obtain optimal order error estimate in the H^1-norm for the bilinear isoparametric interpolation on a convex quadrilateral K, namely, the estimate

$$\|v - v_I\|_0 + h_K|v - v_I|_{1,K} \le Ch_K^2|u|_{2,K}, \ \forall v \in H^1(K), \tag{3.8}$$

we need the following degeneration condition, which was introduced by Acosta and Duran [1].

Definition 3.3. A convex quadrilateral K is said to satisfy the **regular decomposition property** with constants $N \in R$ and $0 < \Psi < \pi$, or shortly $RDP(N, \Psi)$, if we can divide K into two triangles along one of its diagonal, which will always be called d_1, in such a way that $|d_1|/|d_2| \le N$ and both triangles satisfy the maximum angle condition with parameter Ψ(i.e., all angles are bounded by Ψ).

By the definition of J_K, and the geometric relations of (2.3),(3.1)-(3.3), we can easily check that

$$\frac{J_0^K}{J_K} = 1 + O(h_K^\alpha), \ \frac{J_1^K}{J_K} = O(h_K^\alpha), \ \frac{J_2^K}{J_K} = O(h_K^\alpha), \tag{3.9}$$

$$\frac{J_K}{J_0^K} = 1 + O(h_K^\alpha), \ \frac{J_1^K}{J_0^K} = O(h_K^\alpha), \ \frac{J_2^K}{J_0^K} = O(h_K^\alpha), \tag{3.10}$$

Define

$$A_0 = [-a_2, a_1], \ A_1 = [-a_3\xi, a_3\eta]$$
$$B_0 = [b_2, -b_1], \ B_1 = [b_3\xi, -b_3\eta].$$

We denote

$$A = A_0 + A_1, \ B = B_0 + B_1. \tag{3.11}$$

Lemma 3.4. *Let a convex quadrilateral K satisfy the* **diagonal condition**. *Then*

$$(a) \quad \|B_0X^{-1}\|_2 \le 1 + O(h^\alpha), \ \|B_1X^{-1}\|_2 \le O(h^\alpha),$$
$$(b) \quad \|A_0X^{-1}\|_2 \le 1 + O(h^\alpha), \ \|A_1X^{-1}\|_2 \le O(h^\alpha).$$

Proof. We only prove (a). The proof of (b) is similar. It is straightforward to verify that

$$\begin{aligned} B_0X^{-1} &= [b_2, -b_1] \cdot \frac{1}{J_K}(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} + \begin{bmatrix} a_3\eta & b_3\eta \\ a_3\xi & b_3\xi \end{bmatrix}) \\ &= \frac{J_0}{J_K}[1, 0] + \frac{1}{J_K}[b_2a_3 - b_1a_3\xi, b_2b_3\eta - b_1b_3\xi], \end{aligned}$$

$$\begin{aligned} B_1X^{-1} &= [b_3\xi, -b_3\eta] \cdot \frac{1}{J_K}(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} + \begin{bmatrix} a_3\eta & b_3\eta \\ a_3\xi & b_3\xi \end{bmatrix}) \\ &= \frac{1}{J_K}[a_1b_3\xi - a_2b_3\eta, b_3b_1\xi - b_2b_3\eta]. \end{aligned}$$

Then

$$\|B_0X^{-1}\|_2 \le \frac{J_0}{J_K} + C\frac{h^{2+\alpha}}{J_K} \le 1 + O(h^\alpha), \tag{3.12}$$

$$\|B_1X^{-1}\|_2 \le \|\frac{1}{J_K}[a_1b_3\xi - a_2b_3\eta, b_3b_1\xi - b_2b_3\eta]\|_2 \le Ch^\alpha.$$

□

Next we define the finite element space S_h on T_h as follows:

$$S_h = \{v \in H^1(\Omega) : \hat{v} = v \circ F_K \in Q_1(\hat{K}),\ \forall K \in T_h\}, \qquad V_h = S_h \times S_h.$$

Here $Q_1(\hat{K})$ is the bilinear function space on the reference element.

4. **Superconvergence analysis.**

Define $\frac{\tilde{\partial} v}{\partial x_1}$ and $\frac{\tilde{\partial} v}{\partial x_2}$ as follows:

$$\frac{\tilde{\partial} v}{\partial x_1} := \frac{1}{J_0}B_0 \cdot \hat{\nabla}\hat{v},\ \frac{\tilde{\partial} v}{\partial x_2} := \frac{1}{J_0}A_0 \cdot \hat{\nabla}\hat{v}, \tag{4.1}$$

where $B_0 = [b_2, -b_1],\ A_0 = [-a_2, a_1]$. Based on the definition, we can define the modified divergence and strain tensor by

$$\tilde{\text{div}}\mathbf{v} = \frac{\tilde{\partial} v_1}{\partial x_1} + \frac{\tilde{\partial} v_2}{\partial x_2},$$

and

$$\tilde{\varepsilon}(\mathbf{v}) = [\varepsilon_{ij}(\mathbf{v})] = [\frac{1}{2}(\frac{\tilde{\partial} v_i}{\partial x_j} + \frac{\tilde{\partial} v_j}{\partial x_i})],\ i, j = 1, 2,$$

for any $\mathbf{v} = (v_1, v_2)$.

Next, we define some new bilinear forms based on $\frac{\tilde{\partial} v}{\partial x_1}$ and $\frac{\tilde{\partial} v}{\partial x_2}$ on element K as follows:

$$(\varepsilon(\mathbf{w}), \varepsilon(\mathbf{w}))^*_K := J_0^K \int_{\hat{K}} \tilde{\varepsilon}(\mathbf{w}) : \tilde{\varepsilon}(\mathbf{v}) d\xi d\eta,$$

and

$$(\text{div}(\mathbf{w}), \text{div}(\mathbf{v}))^*_K := J_0^K \int_{\hat{K}} \tilde{\text{div}}(\mathbf{w})\tilde{\text{div}}(\mathbf{v}) d\xi d\eta.$$

Theorem 4.1. *Assume that K satisfies the* **diagonal condition**. *Then for any $\mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2) \in V_h$, there exists a constant C independent of mesh sizes h such that*

$$(a) \qquad |(\varepsilon(\mathbf{w}), \varepsilon(\mathbf{v}))_K - (\tilde{\varepsilon}(\mathbf{w}), \tilde{\varepsilon}(\mathbf{v}))^*_K| \le Ch^\alpha \|\mathbf{w}\|_{1,K}\|\mathbf{v}\|_{1,K},$$

$$(b) \qquad |(\text{div}(\mathbf{w}), \text{div}(\mathbf{v}))_K - (\tilde{\text{div}}(\mathbf{w}), \tilde{\text{div}}(\mathbf{v}))^*_K| \le Ch^\alpha \|\mathbf{w}\|_{1,K}\|\mathbf{v}\|_{1,K}.$$

Proof. First we prove (a). The definition of ε, and $\tilde{\varepsilon}$ gives

$$\begin{aligned} & (\varepsilon(\mathbf{w}), \varepsilon(\mathbf{v}))_K - (\tilde{\varepsilon}(\mathbf{w}), \tilde{\varepsilon}(\mathbf{v}))^*_K \\ = & \sum_{i,j=1}^{2} [(\varepsilon_{ij}(\mathbf{w}), \varepsilon_{ij}(\mathbf{v})_K - (\tilde{\varepsilon}_{ij}(\mathbf{w}), \tilde{\varepsilon}_{ij}(\mathbf{v})^*_K]. \end{aligned}$$

We estimate these terms one by one. For the first term, we have

$$
\begin{aligned}
& (\varepsilon_{11}(\mathbf{w}),\varepsilon_{11}(\mathbf{v}))_K-(\tilde\varepsilon_{11}(\mathbf{w}),\tilde\varepsilon_{11}(\mathbf{v}))_K^*\\
= & (\frac{\partial w_1}{\partial x_1},\frac{\partial v_1}{\partial x_1})_K-J_0^K\int_{\hat K}\frac{\tilde\partial w_1}{\partial x_1}\frac{\tilde\partial v_1}{\partial x_1}d\xi d\eta\\
= & \int_{\hat K}(\frac{1}{J_K}B\hat\nabla\hat w_1)^T(\frac{1}{J_K}B\hat\nabla\hat v_1)J_K d\xi d\eta\\
& -J_0^K\int_{\hat K}(\frac{1}{J_0^K}B_0\hat\nabla\hat w_1)^T(\frac{1}{J_0^K}B_0\hat\nabla\hat v_1)d\xi d\eta\\
= & \int_{\hat K}\frac{1}{J_K}(B\hat\nabla\hat w_1)^T(B\hat\nabla\hat v_1)d\xi d\eta-\int_{\hat K}\frac{1}{J_0^K}(B_0\hat\nabla\hat w_1)^T(B_0\hat\nabla\hat v_1)d\xi d\eta\\
= & [\int_{\hat K}\frac{1}{J_K}(B\hat\nabla\hat w_1)^T(B\hat\nabla\hat v_1)d\xi d\eta-\int_{\hat K}\frac{1}{J_0^K}(B\hat\nabla\hat w_1)^T(B\hat\nabla\hat v_1)d\xi d\eta]\\
& +[\int_{\hat K}\frac{1}{J_0^K}(B\hat\nabla\hat w_1)^T(B\hat\nabla\hat v_1)d\xi d\eta-\int_{\hat K}\frac{1}{J_0^K}(B_0\hat\nabla\hat w_1)^T(B_0\hat\nabla\hat v_1)d\xi d\eta]\\
:= & E_1+E_2
\end{aligned}
$$

For the term E_1, we can estimate it as follows:

$$
\begin{aligned}
E_1 &= \int_K\frac{\partial w_1}{\partial x_1}\frac{\partial v_1}{\partial x_1}dx_1dx_2-\int_K\frac{J_K}{J_0^K}\frac{\partial w_1}{\partial x_1}\frac{\partial v_1}{\partial x_1}dx_1dx_2\\
&= \int_K(1-\frac{J_K}{J_0^K})\frac{\partial w_1}{\partial x_1}\frac{\partial v_1}{\partial x_1}dx_1dx_2\\
&= \int_K(\frac{J_1^K}{J_0^K}\xi+\frac{J_2^K}{J_0^K}\eta)\frac{\partial w_1}{\partial x_1}\frac{\partial v_1}{\partial x_1}dx_1dx_2\\
&\le Ch^\alpha|w_1|_1|v_1|_1.
\end{aligned}
$$

In the last inequality, we have used (3.10).

Next, we estimate the term E_2. Because $B=B_0+B_1$, we have

$$
\begin{aligned}
E_2 = & \frac{1}{J_0^K}[\int_{\hat K}(B_1\hat\nabla\hat w_1)^T(B_0\hat\nabla\hat v_1)d\xi d\eta+\int_{\hat K}(B_0\hat\nabla\hat w_1)^T(B_1\hat\nabla\hat v_1)d\xi d\eta\\
& +\int_{\hat K}(B_1\hat\nabla\hat w_1)^T(B_1\hat\nabla\hat v_1)d\xi d\eta].
\end{aligned}
$$

We only estimate the first term in the above formulation. The estimation of the other two terms is similar. Using Lemma 3.4 and (3.7) yields

$$
\begin{aligned}
& \frac{1}{J_0^K}|\int_{\hat K}(B_1\hat\nabla\hat w_1)^T(B_0\hat\nabla\hat v_1)d\xi d\eta|\\
= & \frac{1}{J_0^K}|\int_{\hat K}(\hat\nabla\hat w_1)^T(B_1^TB_0)(\hat\nabla\hat v_1)d\xi d\eta|\\
= & |\int_{\hat K}\frac{J_K}{J_0^K}(\frac{1}{J_K}X\hat\nabla\hat w_1)^T(X^{-1})^T(B_1^TB_0)X^{-1}(\frac{1}{J_K}X\hat\nabla\hat v_1)dx_1dx_2|\\
= & |\int_{\hat K}\frac{J_K}{J_0^K}(\frac{1}{J_K}X\hat\nabla\hat w_1)^T(B_1X^{-1})^T)(B_0X^{-1})(\frac{1}{J_K}X\hat\nabla\hat v_1)dx_1dx_2|\\
= & |\int_{\hat K}\frac{J_K}{J_0^K}(\nabla w_1)^T(B_1X^{-1})^T)(B_0X^{-1})(\nabla v_1)dx_1dx_2|\\
\le & Ch^\alpha|w_1|_{1,K}|v_1|_{1,K}.
\end{aligned}
$$

Hence, we can obtain

$$(\varepsilon_{11}(\mathbf{w}),\varepsilon_{11}(\mathbf{v}))_K-(\tilde{\varepsilon}_{11}(\mathbf{w}),\tilde{\varepsilon}_{11}(\mathbf{v}))_K^*\le Ch^{\alpha}|w_1|_{1,K}|v_1|_{1,K}. \tag{4.2}$$

Similarly, we have

$$(\varepsilon_{22}(\mathbf{w}),\varepsilon_{22}(\mathbf{v}))_K-(\tilde{\varepsilon}_{22}(\mathbf{w}),\tilde{\varepsilon}_{22}(\mathbf{v}))_K^*\le Ch^{\alpha}|w_2|_{1,K}|v_2|_{1,K}. \tag{4.3}$$

For the term,

$$(\varepsilon_{12}(\mathbf{w}),\varepsilon_{12}(\mathbf{v}))_K-(\tilde{\varepsilon}_{12}(\mathbf{w}),\tilde{\varepsilon}_{12}(\mathbf{v}))_K^*,$$

we can derive that

$$\begin{aligned}
&(\varepsilon_{12}(\mathbf{w}),\varepsilon_{12}(\mathbf{v}))_K-(\tilde{\varepsilon}_{12}(\mathbf{w}),\tilde{\varepsilon}_{12}(\mathbf{v}))_K^*\\
=\;& \frac{1}{4}\int_K(\frac{\partial w_2}{\partial x_1}+\frac{\partial w_1}{\partial v_2})(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial v_2})dx_1dx_2\\
&-J_0^K\frac{1}{4}\int_{\hat{K}}(\frac{\tilde{\partial} w_2}{\partial x_1}+\frac{\tilde{\partial} w_1}{\partial v_2})(\frac{\tilde{\partial} v_2}{\partial x_1}+\frac{\tilde{\partial} v_1}{\partial v_2})d\xi d\eta\\
=\;& \frac{1}{4}[\int_K\frac{\partial w_2}{\partial x_1}\frac{\partial v_1}{\partial x_2}dx_1dx_2-J_0^K\int_{\hat{K}}\frac{\tilde{\partial} w_2}{\partial x_1}\frac{\tilde{\partial} v_1}{\partial x_2}d\xi\eta]\\
&+\frac{1}{4}[\int_K\frac{\partial w_1}{\partial x_2}\frac{\partial v_2}{\partial x_1}dx_1dx_2-J_0^K\int_{\hat{K}}\frac{\tilde{\partial} w_1}{\partial x_2}\frac{\tilde{\partial} v_2}{\partial x_1}d\xi\eta]\\
&+\frac{1}{4}[\int_K\frac{\partial w_2}{\partial x_1}\frac{\partial v_1}{\partial x_2}dx_1dx_2-J_0^K\int_{\hat{K}}\frac{\tilde{\partial} w_2}{\partial x_1}\frac{\tilde{\partial} v_1}{\partial x_2}d\xi\eta]\\
&+\frac{1}{4}[\int_K\frac{\partial w_2}{\partial x_1}\frac{\partial v_2}{\partial x_1}dx_1dx_2-J_0^K\int_{\hat{K}}\frac{\tilde{\partial} w_2}{\partial x_1}\frac{\tilde{\partial} v_2}{\partial x_1}d\xi\eta]\\
:=\;& H_1+H_2+H_3+H_4.
\end{aligned}$$

Using the same techniques as the estimation for (4.2), we can estimate $H_i,(i=1,...,4)$ one by one, and finally we can get

$$\sum_{i=1}^{4}|H_i|\le Ch^{\alpha}|\mathbf{w}|_{1,K}|\mathbf{v}|_{1,K}. \tag{4.4}$$

Then combining above two equalities yields

$$(\varepsilon_{12}(\mathbf{w}),\varepsilon_{12}(\mathbf{v}))_K-(\tilde{\varepsilon}_{12}(\mathbf{w}),\tilde{\varepsilon}_{12}(\mathbf{v}))_K^*\le Ch^{\alpha}|\mathbf{w}|_{1,K}|\mathbf{v}|_{1,K}. \tag{4.5}$$

Similarly, we have

$$(\varepsilon_{21}(\mathbf{w}),\varepsilon_{21}(\mathbf{v}))_K-(\tilde{\varepsilon}_{21}(\mathbf{w}),\tilde{\varepsilon}_{21}(\mathbf{v}))_K^*\le Ch^{\alpha}|\mathbf{w}|_{1,K}|\mathbf{v}|_{1,K}, \tag{4.6}$$

which, together with (4.2),(4.3),(4.5), gives (a).

The proof of (b) is similar to the proof of (a).

□

We then define a modified bilinear form over the finite element space V_h as follows:

$$a_h(\mathbf{u},\mathbf{v})=2\mu\sum_K(\tilde{\varepsilon}(\mathbf{u}),\tilde{\varepsilon}(\mathbf{v}))_K^*+\lambda\sum_K(\tilde{\text{div}}\mathbf{u},\tilde{\text{div}}\mathbf{v})_K^*,\ \forall u,v\in V_h.$$

Theorem 4.2. *Assume the partition T_h satisfy the* **diagonal condition** *and RDP(N,Ψ), and let* $\mathbf{u^I}$ *be the finite element interpolation of the function* $\mathbf{u}=$

$(u_1,u_2)\in \mathbf{H}^3(\Omega)\cap\mathbf{H}_0^1(\Omega)$ *under quadrilateral meshes. Then for any* $\mathbf{v}=(v_1,v_2)\in V_h=S_h\times S_h$,

$$|a_h(\mathbf{u}-\mathbf{u}^\mathbf{I},\mathbf{v})|\le C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega}+h^2|\mathbf{u}|_{3,\Omega})\|\mathbf{v}\|_{1,\Omega}. \tag{4.7}$$

Proof. By the definition, we have

$$a_h(\mathbf{u}-\mathbf{u}^\mathbf{I},\mathbf{v})=2\mu\sum_K(\tilde\varepsilon(\mathbf{u}-\mathbf{u}^\mathbf{I}),\tilde\varepsilon(\mathbf{v}))_K^*+\lambda\sum_K(\tilde{\text{div}}(\mathbf{u}-\mathbf{u}^\mathbf{I}),\tilde{\text{div}}\mathbf{v})_K^*, \tag{4.8}$$

For the first term in the right hand side of the above equality, we have

$$(\tilde\varepsilon(\mathbf{u}-\mathbf{u}^\mathbf{I}),\tilde\varepsilon(\mathbf{v}))_K^*=\sum_{i,j=1}^2(\tilde\varepsilon_{ij}(\mathbf{u}-\mathbf{u}^\mathbf{I}),\tilde\varepsilon_{ij}(\mathbf{v}))_K^*.$$

By simple calculation, we get

$$\begin{aligned}
&(\tilde\varepsilon_{11}(\mathbf{u}-\mathbf{u}^\mathbf{I}),\tilde\varepsilon_{11}(\mathbf{v}))_K^*\\
=\ & J_0^K\int_K\frac{\tilde\partial(u_1-u_1^I)}{\partial x_1}\frac{\tilde\partial v_1}{\partial x_1}dx_1dx_2\\
=\ & \int_{\hat K}\hat\nabla(\hat{u_1}-\hat u_1^I).(\frac{1}{J_0^K}B_0^TB_0).\hat\nabla v_1d\xi d\eta\\
=\ & \frac{b_2^2}{J_0^K}\int_{\hat K}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial\xi}\frac{\partial v_1}{\partial\xi}d\xi d\eta+\frac{b_1^2}{J_0^K}\int_{\hat K}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial\eta}\frac{\partial v_1}{\partial\eta}d\xi d\eta\\
&-[\frac{b_2b_1}{J_0^K}\int_{\hat K}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial\xi}\frac{\partial v_1}{\partial\eta}d\xi d\eta+\frac{b_2b_1}{J_0^K}\int_{\hat K}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial\eta}\frac{\partial v_1}{\partial\xi}d\xi d\eta].
\end{aligned}$$

We estimate the terms on the right hand side of the above equality one by one. For the first term, if $\hat u_1\in P_2(\hat K)$, it is easy to see that

$$\int_{\hat K}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial\xi}\frac{\partial v_1}{\partial\xi}d\xi d\eta=0.$$

So by the Bramble-Hilbert Lemma and the fact $|\frac{b_2^2}{J_0^K}|=O(1)$, we have

$$\begin{aligned}
&|\frac{b_2^2}{J_0^K}\int_{\hat K}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial\xi}\frac{\partial v_1}{\partial\xi}d\xi d\eta|\\
\le\ & C\|D^3\hat u_1\|_{0,\hat K}|\hat v_1|_{1,\hat K}\\
\le\ & C(h_K^{1+\alpha}|u_1|_{2,K}+h_K^2|u_1|_{3,K})|v_1|_{1,K}.
\end{aligned}$$

Similarly,

$$|\frac{b_1^2}{J_0^K}\int_{\hat K}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial\eta}\frac{\partial v_1}{\partial\eta}d\xi d\eta|\le C(h_K^{1+\alpha}|u_1|_{2,K}+h_K^2|u_1|_{3,K})|v_1|_{1,K}.$$

Next we estimate the last term. For any $v_1\in S_h$, we can express

$$\frac{\partial\hat v_1}{\partial\xi}=\frac{\partial\hat v_1}{\partial\xi}(0,0)+\eta\frac{\partial^2\hat v_1}{\partial\xi\partial\eta},\ \ \frac{\partial\hat v_1}{\partial\eta}=\frac{\partial\hat v_1}{\partial\eta}(0,0)+\xi\frac{\partial^2\hat v_1}{\partial\xi\partial\eta},$$

Then we can write

$$
\begin{aligned}
& \frac{b_2 b_1}{J_0^K}\Big[\int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi}\frac{\partial v_1}{\partial \eta} d\xi d\eta + \int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \eta}\frac{\partial v_1}{\partial \xi} d\xi d\eta\Big] \\
= \;& \frac{b_2 b_1}{J_0^K}\Big[\frac{\partial v_1}{\partial \eta}(0,0)\int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi} d\xi d\eta + \frac{\partial v_1}{\partial \xi}(0,0)\int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \eta} d\xi d\eta\Big] \\
& + \frac{b_2 b_1}{J_0^K}\Big[\frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \xi \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi} d\xi d\eta + \frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \eta \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \eta} d\xi d\eta\Big].
\end{aligned}
$$

Since for any $\hat u_1 \in P_2(\hat K)$, it is easy to check that

$$
\frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi} d\xi d\eta = 0, \;\; \frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \eta} d\xi d\eta = 0.
$$

Applying Bramble-Hilbert Lemma gives

$$
\begin{aligned}
& \Big|\frac{b_2 b_1}{J_0^K}\Big[\frac{\partial v_1}{\partial \eta}(0,0)\int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi} d\xi d\eta + \frac{\partial v_1}{\partial \xi}(0,0)\int_{\hat K} \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \eta} d\xi d\eta\Big]\Big| \\
\le \;& C\|D^3 \hat u_1\|_{0,\hat K} |\hat v_1|_{1,\hat K} \le C(h_K^{1+\alpha}|u_1|_{2,K} + h_K^2 |u_1|_{3,K})|v_1|_{1,K}.
\end{aligned}
$$

For the term

$$
\begin{aligned}
& \frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \xi \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi} d\xi d\eta = \frac{1}{2}\frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \frac{\partial(\xi^2-1)}{\partial \xi}\frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi} d\xi d\eta \\
= \;& -\frac{1}{2}\frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} (\xi^2-1)\frac{\partial^2(\hat u_1-\hat u_1^I)}{\partial \xi^2} d\xi d\eta = -\frac{1}{2}\frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} (\xi^2-1)\frac{\partial^2 \hat u_1}{\partial \xi^2} d\xi d\eta \\
= \;& \frac{1}{2}\int_{-1}^{1} (\xi^2-1)(\frac{\partial^2 \hat u_1}{\partial \xi^2}\frac{\partial \hat v}{\partial \xi})(\xi,-1) d\xi - \frac{1}{2}\int_{-1}^{1} (\xi^2-1)(\frac{\partial^2 \hat u_1}{\partial \xi^2})\frac{\partial \hat v}{\partial \xi}(\xi,1) d\xi \\
& + \int_{\hat K} (\xi^2-1)\frac{\partial^3 \hat u_1}{\partial \xi^2 \partial \eta}\frac{\partial \hat v_1}{\partial \xi} d\xi d\eta.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \eta \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \eta} d\xi d\eta \\
= \;& \frac{1}{2}\int_{-1}^{1} (\eta^2-1)(\frac{\partial^2 \hat u_1}{\partial^2 \eta}\frac{\partial \hat v}{\partial \eta})(-1,\eta) d\xi - \frac{1}{2}\int_{-1}^{1} (\eta^2-1)(\frac{\partial^2 \hat u_1}{\partial \eta^2})\frac{\partial \hat v}{\partial \xi}(1,\eta) d\xi \\
& + \int_{\hat K} (\xi^2-1)\frac{\partial^3 \hat u_1}{\partial \xi^2 \partial \eta}\frac{\partial \hat v_1}{\partial \xi} d\xi d\eta.
\end{aligned}
$$

Then combining above two equalities yields

$$
\begin{aligned}
& \frac{b_2 b_1}{J_0}\Big[\frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \xi \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \xi} d\xi d\eta + \frac{\partial^2 v_1}{\partial \xi \partial \eta}\int_{\hat K} \eta \frac{\partial(\hat u_1-\hat u_1^I)}{\partial \eta} d\xi d\eta\Big] \\
= \;& \frac{b_2 b_1}{J_0}\Big\{\sum_{i=1}^{4} |l_i|^2 \int_{l_i} (t(s)^2-1)\frac{\partial^2 u_1}{\partial s^2}\frac{\partial v_1}{\partial s} ds \\
& + \Big[\int_{\hat K} (\xi^2-1)\frac{\partial^3 \hat u_1}{\partial \xi^2 \partial \eta}\frac{\partial \hat v_1}{\partial \xi} d\xi d\eta + \int_{\hat K} (\xi^2-1)\frac{\partial^3 \hat u_1}{\partial \xi^2 \partial \eta}\frac{\partial \hat v_1}{\partial \xi} d\xi d\eta\Big]\Big\} \\
\le \;& \frac{b_2 b_1}{J_0}\sum_{i=1}^{4} |l_i| \int_{l_i} (t(s)^2-1)\frac{\partial^2 u_1}{\partial s^2}\frac{\partial v_1}{\partial s} ds \\
& + C(h_K^{1+\alpha}|u_1|_{2,K} + h_K^2 |u_1|_{3,K})|v_1|_{1,K},
\end{aligned}
$$

here l_i are four edges of K. Using the neighboring condition, for any two elements $K_1,\ K_2$, that shares a common edge, and the trace theorem, we have

$$\begin{aligned} & |(\frac{b_1^{K_1}b_2^{K_1}}{J_0^{K_1}} - \frac{b_1^{K_2}b_2^{K_2}}{J_0^{K_2}})|l|^2 \int_{l_i}(t(s)^2-1)\frac{\partial^2 u_1}{\partial s^2}\frac{\partial v_1}{\partial s}ds \\ \leq\ & Ch^\alpha |l|^2(h^{-1}|u_1|_{2,K}|v_1|_{1,K} + h|u_1|_{3,K}|v_1|_{1,K}) \\ \leq\ & C(h_K^{1+\alpha}|u_1|_{2,K} + h_K^2|u_1|_{3,K})|v_1|_{1,K}. \end{aligned}$$

Adding all elements $K \in T$ together gives

$$\begin{aligned} & \sum_K (\tilde{\varepsilon}_{11}(\mathbf{u}-\mathbf{u^I}), \tilde{\varepsilon}_{11}(\mathbf{v}))_K^* \\ \leq\ & C(h^{1+\alpha}|u_1|_2 + h^2|u_1|_3)|v_1|_1. \end{aligned} \tag{4.9}$$

Similarly, we have

$$\begin{aligned} & \sum_K (\tilde{\varepsilon}_{22}(\mathbf{u}-\mathbf{u^I}), \tilde{\varepsilon}_{22}(\mathbf{v}))_K^* \\ \leq\ & C(h^{1+\alpha}|u_1|_2 + h^2|u_1|_3)|v_1|_1. \end{aligned} \tag{4.10}$$

For the term

$$\begin{aligned} & \sum_K (\tilde{\varepsilon}_{12}(\mathbf{u}-\mathbf{u^I}), \tilde{\varepsilon}_{12}(\mathbf{v}))_K^* \\ =\ & \frac{1}{4}\sum_K \int_K (\frac{\tilde{\partial}(u_2-u_2^I)}{\partial x_1}dx_1dx_2 + \frac{\tilde{\partial}(u_1-u_1^I)}{\partial x_2})(\frac{\tilde{\partial} v_2}{\partial x_1} + \frac{\tilde{\partial} v_1}{\partial x_2})dx_1dx_2 \\ =\ & \frac{1}{4}\sum_K [\int_K (\frac{\tilde{\partial}(u_2-u_2^I)}{\partial x_1}\frac{\tilde{\partial} v_2}{\partial x_1}dx_1dx_2 + \frac{\tilde{\partial}(u_2-u_2^I)}{\partial x_1}\frac{\tilde{\partial} v_1}{\partial x_2}dx_1dx_2 \\ & + \int_K (\frac{\tilde{\partial}(u_1-u_1^I)}{\partial x_2}\frac{\tilde{\partial} v_2}{\partial x_1}dx_1dx_2 + \frac{\tilde{\partial}(u_1-u_1^I)}{\partial x_1}\frac{\tilde{\partial} v_1}{\partial x_2}dx_1dx_2] \\ :=\ & I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using same arguments as in the proof of (4.9), we can estimate I_i one by one, and finally we can get

$$\begin{aligned} & \sum_K (\tilde{\varepsilon}_{12}(\mathbf{u}-\mathbf{u^I}), \tilde{\varepsilon}_{12}(\mathbf{v}))_K^* \\ \leq\ & C(h^{1+\alpha}|\mathbf{u}|_2 + h^2|\mathbf{u}|_3)|\mathbf{v}|_1. \end{aligned} \tag{4.11}$$

Similarly, we have

$$\begin{aligned} & \sum_K (\tilde{\varepsilon}_{21}(\mathbf{u}-\mathbf{u^I}), \tilde{\varepsilon}_{21}(\mathbf{v}))_K^* \\ \leq\ & C(h^{1+\alpha}|\mathbf{u}|_2 + h^2|\mathbf{u}|_3)|\mathbf{v}|_1. \end{aligned} \tag{4.12}$$

Using the same technique, we can prove that

$$\begin{aligned} & \sum_K (\tilde{\text{div}}(\mathbf{u}-\mathbf{u^I}), \tilde{\text{div}}\mathbf{v})_K^* \\ \leq\ & C(h^{1+\alpha}|\mathbf{u}|_2 + h^2|\mathbf{u}|_3)|\mathbf{v}|_1. \end{aligned}$$

Conclusion follows by combining (4.8)-(4.12). □

From the proof of the above theorem, we know that the following result is also valid.

Proposition 4.1. *Assume the partition T_h satisfy the* **diagonal condition** *and RDP(N,Ψ), and let $\mathbf{u^I}$ be the finite element interpolation of the function $\mathbf{u} \in \mathbf{H}^3(\Omega)$, under the quadrilateral meshes. Then for any $\Omega_0 \subset\subset \Omega$, $\mathbf{v} \in V_h = S_h \times S_h$, we have*

$$|a_{h,\Omega_0}(\mathbf{u} - \mathbf{u^I}, \mathbf{v})| \le C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega_0} + h^2|\mathbf{u}|_{3,\Omega_0})\|\mathbf{v}\|_{1,\Omega}. \tag{4.13}$$

Theorem 4.3. *Assume that T_h satisfies* **diagonal condition***, and RDP(N, Ψ). Let $\mathbf{u} \in \mathbf{H}^3(\Omega) \cap \mathbf{H}^1_0(\Omega)$ be the solution of (2.1), let $\mathbf{u}_h$ and $\mathbf{u}^I$ be the finite element approximation and the finite element interpolation of* $\mathbf{u}$, *respectively. Then*

$$\begin{aligned} &(1) \qquad |\mathbf{u^I} - \mathbf{u}_h|_1 \le C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega} + h^2|\mathbf{u}|_{3,\Omega}) \\ &(2) \qquad \|\sigma(\mathbf{u^I}) - \sigma(\mathbf{u}_h)\|_0 \le C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega} + h^2|\mathbf{u}|_{3,\Omega}). \end{aligned}$$

Proof. By Theorem 4.1, and Theorem 4.2, we can derive

$$\begin{aligned} a(\mathbf{u} - \mathbf{u^I}, \mathbf{v}) &\le |a(\mathbf{u} - \mathbf{u^I}, \mathbf{v}) - a_h(\mathbf{u} - \mathbf{u^I}, \mathbf{v})| + |a_h(\mathbf{u} - \mathbf{u^I}, \mathbf{v})| \\ &\le Ch^\alpha \sum_K |\mathbf{u} - \mathbf{u^I}|_{1,K}|\mathbf{v}|_1 + a_h(\mathbf{u} - \mathbf{u^I}, \mathbf{v}) \\ &\le C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega} + h^2|\mathbf{u}|_{3,\Omega})|\mathbf{v}|_1. \end{aligned}$$

Using Korn equality gives

$$\begin{aligned} |\mathbf{u^I} - \mathbf{u}_h|_1 &\le C|||\mathbf{u^I} - \mathbf{u}_h||| \\ &\le C \sup_{\mathbf{v} \in \mathbf{V_h}} \frac{a(\mathbf{u^I} - \mathbf{u}_h, \mathbf{v})}{|||\mathbf{v}|||} = \sup_{\mathbf{v} \in \mathbf{V_h}} \frac{a(\mathbf{u^I} - \mathbf{u}, \mathbf{v})}{|||\mathbf{v}|||} \\ &\le C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega} + h^2|\mathbf{u}|_{3,\Omega}), \end{aligned}$$

where $|||\cdot||| = a(\cdot,\cdot)^{\frac{1}{2}}$. By the definition of the stress tensor, we have

$$\begin{aligned} \|\sigma(\mathbf{u^I}) - \sigma(\mathbf{u_h})\|_0 &\le 2\mu\|\varepsilon(\mathbf{u^I}) - \varepsilon(\mathbf{u}_h)\|_0 + \lambda\|\text{div}\mathbf{u^I} - \text{div}\mathbf{u}_h\|_0 \\ &\le C|\mathbf{u^I} - \mathbf{u}_h|_1 \le C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega} + h^2|\mathbf{u}|_{3,\Omega}). \end{aligned}$$

We complete the proof. □

5. **Stress tensor recovery.** In this section, we will introduce some patch recovery methods, which were proposed in [23],[17] for the second order elliptic problems. First we define a gradient recovery operator $G_h : S_h \to S_h \times S_h$, where S_h is the bilinear element space under quadrilateral meshes, as follows: Given the finite element function v_h, we first define $G_h v_h$ at all vertices, and then obtain $G_h v_h$ on the whole domain by interpolation using the original nodal shape functions of S_h.

Given an interior vertex $\mathbf{z}_i$, we select an element patch ω_i, where

$$\bar{\omega}_i = \cup_{K \in T_h, \mathbf{z} \in \bar{K}} \bar{K}.$$

We denote all nodes on $\bar{\omega}_i$(including $\mathbf{z}_i$) as $\mathbf{z}_{ij}, j = 1, 2, ..., n(\ge 6)$. In the following, we will introduce several fitting methods to recover the gradient of a function $v_h \in S_h$ at $\mathbf{z}_i$ by values of $\mathbf{z}_{ij}$. To demonstrate the idea, we write the recovery procedure explicitly on the element patch shown by Fig. 2. For general quadrilateral meshes, a computer algorithm will be needed. We use local coordinates (x_1, x_2) with $\mathbf{z}_i$ as the origin.

1) Fitting

$$p_1^{x_1}(x_1,x_2)=\vec{p}^T\vec{a}=(1,x_1,x_2)(a_1,a_2,a_3)^T$$

with respect to the four derivative values at the center of each element on the patch results in $Q^TQ\vec{a}=Q^T\vec{\sigma}$ where $\vec{\sigma}^T=(\sigma_1,\cdots,\sigma_4)$ (here we use σ to represent $\partial_{x_1}v_h$ since the case for $\partial_{x_2}v_h$ is the same) and

$$Q^T=\begin{pmatrix}1 & 1 & 1 & 1\\ -1/2 & 1/2 & 1/2 & -1/2\\ -1/2 & -1/2 & 1/2 & 1/2\end{pmatrix}.$$

It is easy to calculate

$$Q^TQ=\begin{pmatrix}4 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Therefore,

$$p_1^{x_1}(x_1,x_2)=\frac{1}{4}\sum_{j=1}^{4}\sigma_j+\frac{1}{2}(-\sigma_1+\sigma_2+\sigma_3-\sigma_4)x_1+\frac{1}{2}(-\sigma_1-\sigma_2+\sigma_3+\sigma_4)x_2.$$

We then have

$$p_1^{x_1}(0,0)=\frac{1}{8h}[2(v_1-v_5)+v_2-v_4+v_8-v_6].$$

Similarly,

$$p_1^{x_2}(0,0)=\frac{1}{8h}[2(v_3-v_7)+v_2-v_8+v_4-v_6],$$

where $v_i,(i=1,...,8)$ denote the values of the function v_h at the vertices of $K\in\omega_{\mathbf{z}}$ (cf. Figure 2 for details).

Now define

$$G_hv_h(\mathbf{z}_i)=(G_h^1v_h(\mathbf{z}_i),G_h^2v_h(\mathbf{z}_i))^T=(p_1^{x_1}(0,0),p_1^{x_2}(0,0))^T.$$

2) Fitting

$$q_1^{x_1}(x_1,x_2)=\vec{p}^T\vec{a}=(1,x_1,x_2,x_1x_2)(a_1,a_2,a_3,a_4)^T$$

with respect to the four gradient values at the center of each element on the patch is the same as the interpolation. Therefore,

$$\begin{aligned}q_1^{x_1}(x_1,x_2) &= \frac{1}{4}\sum_{j=1}^{4}\sigma_j+\frac{1}{2}(-\sigma_1+\sigma_2+\sigma_3-\sigma_4)x_1+\frac{1}{2}(-\sigma_1-\sigma_2+\sigma_3+\sigma_4)x_2\\ &\quad +(\sigma_1-\sigma_2+\sigma_3-\sigma_4)x_1x_2\\ &= p_1(x_1,x_2)+(\sigma_1-\sigma_2+\sigma_3-\sigma_4)x_1x_2.\end{aligned}$$

Note that $q_1^{x_1}(0,0)=p_1^{x_1}(0,0)$, the same as in 1). Similarly, we have $q_2^{x_1}(0,0)=p_2^{x_1}(0,0)$. Then we define

$$G_hv_h(\mathbf{z}_i)=(G_h^1v_h(\mathbf{z}_i),G_h^2v_h(\mathbf{z}_i))^T=(q_1^{x_1}(0,0),q_1^{x_2}(0,0))^T=(p_1^{x_1}(0,0),p_1^{x_2}(0,0))^T.$$

3) Fitting

$$p_2(x_1,x_2)=(1,x_1,x_2,x_1^2,x_1x_2,x_2^2)(a_1,\cdots,a_6)^T$$

with respect to the nine nodal values on the patch. Now

$$\vec{e}=(1,1,1,1,1,1,1,1,1)^T,\quad \vec{x}=(0,1,1,0,-1,-1,-1,0,1)^T,$$

$$\vec{y}=(0,0,1,1,1,0,-1,-1,-1)^T,\quad A=(\vec{e},\vec{x},\vec{y},\vec{x}^2,\vec{xy},\vec{y}^2),$$

$$(Q^TQ)^{-1}Q^T = \mathrm{diag}(\frac{1}{9},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{4},\frac{1}{6})\cdot \begin{pmatrix} 5 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1\\ 0 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 1\\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1\\ -2 & 1 & 1 & -2 & 1 & 1 & 1 & -2 & 1\\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1\\ -2 & -2 & 1 & 1 & 1 & -2 & 1 & 1 & 1 \end{pmatrix}.$$

$$\frac{\partial p_2}{\partial x_1}(0,0) = \frac{1}{6h}(v_1 - v_5 + v_2 - v_4 + v_8 - v_6),$$

and

$$\frac{\partial p_2}{\partial x_2}(0,0) = \frac{1}{6h}(v_2 - v_8 + v_3 - v_7 + v_4 - v_6).$$

Now define

$$G_h v_h(\mathbf{z}_i) = (G_h^1 v_h(\mathbf{z}_i), G_h^2 v_h(\mathbf{z}_i))^T = \nabla p_2(0,0;\mathbf{z}_i).$$

4) Fitting

$$q_2(x_1,x_2) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^2x_2, x_1x_2^2, x_1^2x_2^2)(a_1,\cdots,a_9)^T$$

with respect to the nine nodal values on the patch is the same as interpolation. This results in

$$\frac{\partial q_2}{\partial x_1}(0,0) = \frac{1}{2h}(v_1 - v_5), \tag{5.1}$$

and

$$\frac{\partial q_2}{\partial x_2}(0,0) = \frac{1}{2h}(v_3 - v_7), \tag{5.2}$$

which is different from all above. Fitting

$$\tilde{q}_2(x_1,x_2) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^2x_2, x_1x_2^2)(a_1,\cdots,a_8)^T$$

with respect to the nine nodal values on the patch also produces (5.1),(5.2).

Next define

$$G_h v_h(\mathbf{z}_i) = (G_h^1 v_h(\mathbf{z}_i), G_h^2 v_h(\mathbf{z}_i))^T = \nabla q_2(0,0;\mathbf{z}_i).$$

Remark 4. Fitting strategies 1) & 2) are the well known ZZ patch recovery, and strategies 3) & 4) were proposed by Zhang and Naga in [17].

Then for any $\mathbf{u}_h = (u_h^1, u_h^2)$, we define the Stress Tensor Recovery as follows

$$\sigma_h^*(\mathbf{u}_h) = 2\mu\varepsilon^*(\mathbf{u}_h) + \lambda \mathrm{div}^*(\mathbf{u}_h), \tag{5.3}$$

where

$$\varepsilon^*(\mathbf{u}_h) = \sum_{i,j=1}^{2} \frac{1}{2}(G_h^j u_h^i + G_h^i u_h^j),$$

and

$$\mathrm{div}^\star(\mathbf{u}_h) = G_h^1 u_h^1 + G_h^2 u_h^2,$$

where $G_h^1,\ G_h^2$ are defined by the above four fitting methods.

When we consider the pure traction problem, we do not need to do stress tensor recovery on the boundary. However, if we consider the pure displacement boundary value problem, the recovered strain tensor on a boundary node $\mathbf{z}$ can be determined from an element patch ω_i such that $\mathbf{z} \in \omega_i$ in the following way: As an example, we

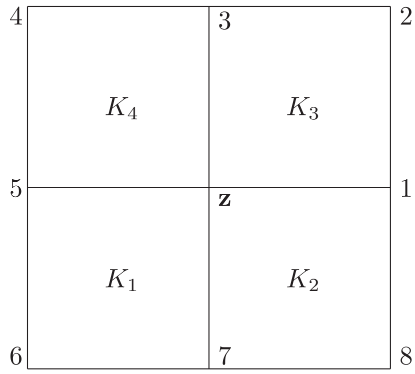

FIGURE 2. Element Patch $\omega_{\mathbf{z}}$

consider the fitting strategy 3). Let the relative coordinates of $\mathbf{z}$ with respect to $\mathbf{z}_i$ is (h, h), then

$$G_h v_h(\mathbf{z}_i) = (G_h^1 v_h(\mathbf{z}_i), G_h^2 v_h(\mathbf{z}_i))^T = \nabla p_2(h, h; \mathbf{z}_i).$$

Putting the gradient recovery to the (5.3) yields the stress tensor recovery. If $\mathbf{z}_i$ is covered by more than one element patches, then some averaging may be applied (cf. [17] for details).

Theorem 5.1. *Let T_h satisfy the* **diagonal condition**. *Then there exists a constant C independent of h such that*

$$\|\sigma^*(\mathbf{v}_h)\|_0 \le C|\mathbf{v}_h|_1 \;\forall \mathbf{v}_h \in V_h.$$

Proof. We first consider an element patch that contains four uniform square elements (Figure 2).

We discuss only the fitting strategy 3), the proof for the another three strategies is similar. Following the recovery procedure of the above, it is straightforward to derive

$$G_h^1 v(\mathbf{z}) = \frac{1}{h}\frac{\partial}{\partial \xi}\hat{p}_2^v(0,0) = \frac{1}{6h}(v_1 - v_5 + v_2 - v_4 + v_8 - v_6) = \frac{1}{h}\sum_j \bar{c}_j^1 v_j, \tag{5.4}$$

where the weighted c_j^1 are the second row of $(Q_0^T Q_0)^{-1} Q_0^T$ for the x_1-derivative and similarly

$$G_h^2 v(\mathbf{z}) = \frac{1}{h}\frac{\partial}{\partial \eta}\hat{p}_2^v(0,0) = \frac{1}{6h}(v_2 - v_8 + v_3 - v_7 + v_4 - v_6) = \frac{1}{h}\sum_j \bar{c}_j^2 v_j, \tag{5.5}$$

where the weighted c_j^2 are the third row of $(Q_0^T Q_0)^{-1} Q_0^T$ for the x_2-derivative. By inverse inequality,

$$\begin{aligned} |v_1 - v_5|^2 &\le 2|v_1 - v(\mathbf{z})|^2 + 2|v(\mathbf{z}) - v_5|^2 \\ &\le Ch^2(|v|^2_{1,\infty,K_2} + |v|^2_{1,\infty,K_1}) \\ &\le C(|v|^2_{1,K_2} + |v|^2_{1,K_1}) \\ &= C|v|^2_{1,K_1\cup K_2}. \end{aligned}$$

Similarly

$$|v_2 - v_4|^2 \le C|v|^2_{1,K_3\cup K_4}, \; |v_8 - v_6|^2 \le C|v|^2_{1,K_1\cup K_2},$$

Then we have

$$\begin{aligned} G_h^1 v(\mathbf{z}) &\leq Ch^{-1}(|v_1 - v_5| + |v_2 - v_4| + |v_8 - v_6|) \\ &\leq Ch^{-1}(|v_1 - v_5|^2 + |v_2 - v_4|^2 + |v_8 - v_6|^2)^{\frac{1}{2}} \\ &\leq Ch^{-1}|v|_{1,\omega_{\mathbf{z}}}. \end{aligned}$$

We can also get

$$G_h^2 v(\mathbf{z}) \leq Ch^{-1}|v|_{1,\omega_{\mathbf{z}}}.$$

By linear mapping, these results are also valid for four uniform parallelogram.

We observe that the diagonal condition together with the neighboring condition imply that for any given node $\mathbf{z}_i$, there are four elements attached to it when h is sufficiently small. In addition, we can decompose the least square fitting matrix Q associated with those four quadrilateral elements as follows:

$$Q = Q_0 + h^\alpha Q_1$$

where Q_0 is the least-square fitting matrix associated with those four parallelograms. After some simple calculations, we can show that

$$(Q^T Q)^{-1} Q^T = (Q_0^T Q_0)^{-1} Q_0^T + O(h^\alpha) I,$$

where I is identity matrix.

Then

$$G_h^1 v(\mathbf{z}) = \frac{1}{h} \sum_j (\vec{c}_j^1 + O(h^\alpha)) v_j.$$

Notice that $\vec{c}$ is constant vector, so for sufficiently small h, we always have

$$|G_h^1 v(\mathbf{z})| \leq \frac{1}{h} \sum_j (\vec{c}_j^1) v_j \leq Ch^{-1}|v|_{1,\omega_{\mathbf{z}}}. \tag{5.6}$$

Then for any $K \in T_h$, define

$$\bar{\omega}_K = \cup_{K' \in T_h, \bar{K}' \cap \bar{K}} \bar{K}'.$$

Let $\mathbf{z}_i,\ i = 1, 2, 3, 4$ be the four vertices of the element K, by scaling argument, it is straightforward to show that

$$\|G_h^1 v\|_{0,K}^2 \leq Ch_K^2 \sum_{i=1}^4 [G_h^1 v(\mathbf{z}_i)]^2.$$

Combining above two equalities and summing all $K \in T_h$ together yields

$$\begin{aligned} \|G_h^1 v\|_0^2 &= \sum_K \|G_h^1 v\|_{0,K}^2 \\ &\leq \sum_K h_K^2 \sum_{i=1}^4 [G_h^1 v(\mathbf{z}_i)]^2 \leq Ch^2 \cdot h^{-2} \sum_K |v|_{1,\omega_K}^2, \end{aligned}$$

which means

$$\|G_h^1 v\|_0 \leq C|v|_1.$$

Similarly,

$$\|G_h^2 v\|_0 \leq C|v|_1.$$

By the definition of the recovery stress tensor and using the above two equalities, we see that

$$\|\sigma^*(\mathbf{v}_h)\|_0 \leq C|\mathbf{v}_h|_1,\ \forall \mathbf{v}_h \in V_h.$$

We complete the proof. □

Remark 5. Another important feature of the recovery operator by the fitting strategies 3) & 4) is the polynomial preserving property, which have already been observed in [17],[19]. More precisely, let $K \in T$, and $\mathbf{u}$ be a quadratic polynomial on ω_K. Assume that K and all elements adjacent to K are convex. Then

$$\sigma^*(\mathbf{u}) = \sigma(\mathbf{u}), \tag{5.7}$$

where $\mathbf{u} = (u_1, u_2)$. Actually, the fitted polynomial $p_2 = u_i$ when u_i $(i = 1, 2)$ is a quadratic polynomial [17]. As a consequence, $\nabla p_2 = \nabla u_i$, $i = 1, 2$.

Theorem 5.2. *Let T_h satisfy the* **diagonal condition** *and $RDP(N, \Psi)$. Let $\mathbf{u} \in \mathbf{H}^3(\Omega) \cap \mathbf{H}_0^1(\Omega)$ be the solution of (2.1), and let $\mathbf{u}_h \in S_h$ be the finite element approximation of* $\mathbf{u}$. *Then the recovered Stress Tensor is superconvergent in the sense*

$$\|\sigma(\mathbf{u}) - \sigma^*(\mathbf{u}_h)\|_0 \le C(h^{1+\alpha}|\mathbf{u}|_2 + h^2|\mathbf{u}|_3),$$

where C is constant independent of $\mathbf{u}$ *and h.*

Proof. We first consider the fitting strategies 3) & 4). We can decompose the error into

$$\sigma(\mathbf{u}) - \sigma^*(\mathbf{u}_h) = \sigma(\mathbf{u}) - \sigma^*(\mathbf{u}) + \sigma^*(\mathbf{u}^{\mathbf{I}} - \mathbf{u}_h), \tag{5.8}$$

here we have used the fact $\sigma^*(\mathbf{u}) = \sigma^*(\mathbf{u}^{\mathbf{I}})$. By the polynomial preserving property for the fitting strategies 3) & 4), and the Bramble-Hilbert Lemma, we can get

$$\|\sigma(\mathbf{u}) - \sigma^*(\mathbf{u})\|_0 \le Ch^2|\mathbf{u}|_3. \tag{5.9}$$

For the fitting strategies 1) & 2), we continue to split the term $\sigma(\mathbf{u}) - \sigma^*(\mathbf{u})$ as

$$\sigma(\mathbf{u}) - \sigma^*(\mathbf{u}) = \sigma(\mathbf{u}) - [\sigma(\mathbf{u})]^I + [\sigma(\mathbf{u})]^I - \sigma^*(\mathbf{u}),$$

where $[\sigma(\mathbf{u})]^I$ is the bilinear interpolation.
By the interpolation estimate, we have

$$\|\sigma(\mathbf{u}) - [\sigma(\mathbf{u})]^I\|_0 \le Ch^2|\mathbf{u}|_3.$$

Moreover, using the similar techniques developed by [15], [19], we can prove that

$$\|[\sigma(\mathbf{u})]^I - \sigma^*(\mathbf{u})\|_0 \le Ch^2|\mathbf{u}|_3.$$

So for the strategies 1) & 2), we also have

$$\|\sigma(\mathbf{u}) - \sigma^*(\mathbf{u})\|_0 \le Ch^2|\mathbf{u}|_3. \tag{5.10}$$

On the other hand, by Theorems 4.2 and 5.1, we can derive

$$\|\sigma^*(\mathbf{u}^{\mathbf{I}} - \mathbf{u}_h)\|_0 \le |\mathbf{u}^{\mathbf{I}} - \mathbf{u}_I|_1 \le C(h^{1+\alpha}|\mathbf{u}|_2 + h^2|\mathbf{u}|_3), \tag{5.11}$$

which, combining (5.8)–(5.10), yields Theorem 5.2. □

For the pure traction problem, we can consider so-called interior estimation, which was first investigated by Nitsche and Schatz [12]. Using this technique, the global regularity $u \in \mathbf{H}^3(\Omega)$, which may not hold in general, is not required. In the following, we will provide a local result based on the interior estimate.

We consider $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ where Ω_0 and Ω_1 are compact polygonal subdomains that can be decomposed into quadrilaterals. Here "compact subdomains" means that $dis(\Omega_0, \partial\Omega_1)$ and $dis(\Omega_1, \partial\Omega)$ are of order $O(1)$. Combining the results developed in Section 4 and the standard interior estimate technique in [12], we immediately obtain the following result.

Theorem 5.3. *Let $\Omega \subset R^2$ be a polygonal domain and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. Assume that T_h satisfy the diagonal condition and $RDP(N, \Psi)$ on Ω_1. Let $\mathbf{u} \in \mathbf{H}^3(\Omega) \cap \mathbf{H}_0^1(\Omega)$ be the solution of (2.1), and let $\mathbf{u}_h \in S_h$ be the finite element approximation of $\mathbf{u}$. Then*

$$\|\sigma(\mathbf{u}) - \sigma^*(\mathbf{u}_h)\|_{0,\Omega_0} \leq C(h^{1+\alpha}|\mathbf{u}|_{2,\Omega} + h^2|\mathbf{u}|_{3,\Omega_0}),$$

where C is constant independent of $\mathbf{u}$ and h.

6. **A posterior error estimate.** Let $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$, our purpose is to estimate the error $\|\sigma(\mathbf{e}_h)\|_{0,\Omega_0}$ by a computable quantity η_h. We can define the error estimator by the recovered stress tensor

$$\eta_h = \|\sigma^*(\mathbf{u}_h) - \sigma(\mathbf{u}_h)\|_{0,\Omega_0}.$$

We assume that

$$\|\sigma(\mathbf{e}_h)\|_{0,\Omega_0} \geq Ch. \tag{6.1}$$

Theorem 6.1. *Assume that the same hypotheses as in Theorem 5.3. Let (6.1) be satisfied. Then*

$$\frac{\eta_h}{\|\sigma(\mathbf{e}_h)\|_{0,\Omega_0}} = 1 + O(h^\rho),\ \rho = min(1, \alpha). \tag{6.2}$$

Proof. By the triangle inequality, we have

$$\eta_h - \|\sigma(\mathbf{u}) - \sigma^*(\mathbf{u}_h)\|_{0,\Omega_0} \leq \|\sigma(\mathbf{e}_h)\|_{0,\Omega_0} \leq \eta_h + \|\sigma(\mathbf{u}) - \sigma^*(\mathbf{u}_h)\|_{0,\Omega_0}.$$

Divided the above inequality by $\|\sigma(\mathbf{e}_h)\|_{0,\Omega_0}$ yields (6.2). □

Remark 6. In this paper, we just consider the compressible elasticity. For the incompressible elasticity, i.e., Poisson ratio $\nu \to \frac{1}{2}$, or equivalently $\lambda \to \infty$, we can use the so-called enhanced strain finite element method [13] to cope with it. The advantage of this method is that it provides a stable locking-free element without the need of implementing the filter. Another important feature of this method is that we only need to solve a symmetric and definite algebraic system. The saddle point algebraic system will be avoided. Along this line, some previous results on non-conforming finite element methods for elasticity [6, 10, 18] would be useful.

Remark 7. Generalization of our results to higher order quadrilateral elements is feasible. Mesh conditions would be more important since for non-tensor product space, degeneration of approximation order may happen [3].

Acknowledgements. We would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] G. Acosta and R. G. Duran, *Error estimates for Q_1 isoparameter element satisfying a weak angle condition*, SIAM J. Numer. Anal., **38** (2001), 1073–1088.

[2] M. Ainsworth and J. T. Oden, "Aa posteriori Error Estimation in Finite Element Analysis," Wiley Interscience, New York, 2000.

[3] D. N. Arnold, D. Boffi and R. S. Falk, *Approximation by quadrilateral finite elements*, Math. Comp., **71** (2002), 909–922.

[4] I. Babuška and T. Strouboulis, "The Finite Element Method and its Reliability," Oxford University Press, London, 2001.

[5] S. C. Brenner and L. R. Scott, "The Mathematical Theory of Finite Element Methods," Springer-Verlag, New York, 1994.

[6] S. C. Brenner and L.-Y. Sung, *Linear finite element methods for planar linear elasticity*, Math. Comp., **59** (1992), 321–338.

[7] C. Carstensen and S. A. Funken, *Averaging technique for FE-a posteriori error control in elasticity. Part I: Conforming FEM*, Comput. Methods Appl. Mech. Engrg., **190** (2001), 2483–2498.

[8] C. Carstensen and S. A. Funken, *Averaging technique for FE-a posteriori error control in elasticity. Part II: λ-independent estimates*, Comput. Methods Appl. Mech. Engrg., **190** (2001), 4663–4675.

[9] P. G. Ciarlet, "The Finite Element Method for Elliptic Problems," North-Holland, Amsterdam, 1978.

[10] R. S. Falk, *Nonconforming finite element methods for the equations of linear elasticity*, Math. Comp., **57** (1991), 529–550.

[11] P. B. Ming and Z.-C. Shi, *Quadrilatral mesh revisited*, Comput. Methods Appl. Mech. Engrg., **191** (2002), 5671–5682.

[12] J. A. Nitsche and A. H. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp., **28** (1974), 937–958.

[13] B. D. Reddy and J. C. Simo, *Stability and convergence of a class of enhanced strain finite element methods*, SIAM J. Numer. Anal., **32** (1995), 1705–1728.

[14] Z.-C. Shi, *A convergence condition for the quadrilateral Wilson element*, Numer. Math., **44** (1984), 349–361.

[15] J. Xu and Z. Zhang, *Analysis od recovery type a posteriori error estimates for mildly structured grids*, Math. Comp., **73** (2003), 1139–1152.

[16] S. Zhang, *On the nested refinement of quadrilateral and hexahedral finite elements and the affine approximation*, Numer. Math., **98** (2004), 559–579.

[17] Z. Zhang and A. Naga, *A new finite element gradient recovery method: superconvergence property*, SIAM J. Sci. Comput., **26** (2005), 1192–1213.

[18] Z. Zhang, *Ananlysis of some quadrilateral nonconforming elements for incompressible elasticity*, SIAM J. Numer. Anal., **34** (1997), 640–663.

[19] Z. Zhang, *Polynomial preserving recovery for anisotropic and irregular grids*, J. Comput. Math., **22** (2004), 331–340.

[20] Z. Zhang, *Polynomail preserving gradient recorery and a posteriori estimate for bilinear element on irregular quadrilaterals*, Inter. J. Numer. Anal. Model., **1** (2004), 1–24.

[21] O. C. Zienkiewicz, R. L. Taylor and J. Z. Zhu, "The Finite Element Method," 6th ed., McGraw-Hill, London, 2005.

[22] O. C. Zienkiewicz and J. Z. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis*, Int. J. Numer. Methods Engrg., **24** (1987), 337–357.

[23] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergence patch recovery and a posteriori error estimates. Part 1: The recovery technique*, Int. J. Numer. Methods Engrg., **33** (1992), 1365–1382.

[24] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergence patch recovery and a posteriori error estimates. Part 2: Error estimates and adaptivity*, Int. J. Numer. Methods Engrg., **33** (1992), 1365–1382.

Received January 2007; revised September 2007.

E-mail address: zzhang@math.wayne.edu