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Superconvergence of a Chebyshev Spectral Collocation Method

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Abstract We reveal the relationship between a Petrov–Galerkin method and a spectral collocation method at the Chebyshev points of the second kind (± 1 and zeros of U_k) for the two-point boundary value problem. Derivative superconvergence points are identified as the Chebyshev points of the first kind (Zeros of T_k). Super-geometric convergent rate is established for a special class of solutions.

Keywords Chebyshev polynomials · Collocation · Spectral method · Superconvergence · Petrov–Galerkin

1 Introduction

Perhaps the most attractive feature of the spectral method [2–5, 8–10, 12, 15, 16] is its geometric convergence rate under analytic assumption for the exact solution. In [18], a super-geometric convergence rate was established for a spectral collocation method using the Gauss points for two-point boundary value problems. The method was interpreted as a spectral Galerkin method up to some higher-order numerical integration errors. Naturally, we would want to know if a similar result can be proved for collocation at the Chebyshev points. Unfortunately, this generalization is not straightforward.

Collocation at the Legendre–Gauss–Lobatto points can be naturally linked to a symmetric Galerkin method via the Legendre–Gauss–Lobatto quadrature rule, while collocation at the Chebyshev–Gauss–Lobatto points can only be linked to a non-symmetric (or weighted) Galerkin method via the Chebyshev–Gauss–Lobatto quadrature rule. It turns out that this link is a Petrov–Galerkin method, i.e., different trail and test spaces are used.

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We start from an observation to the Chebyshev polynomials: if $f_I \in \mathcal{P}_p$ interpolates f at the following set of the Chebyshev points

$$x_j = \cos\theta_j, \quad \theta_j = \frac{\pi j}{p}, \quad j = 0, 1, \dots, p, \tag{1.1}$$

then the remainder can be expressed as

$$\begin{aligned} f(x) - f_I(x) &= f[x_0, x_1, \dots, x_p, x]\frac{1}{2^{p-1}p}(x^2-1)T_p'(x) \\ &= f[x_0, x_1, \dots, x_p, x]\frac{1}{2^{p-1}}(x^2-1)U_{p-1}(x) \\ &= f[x_0, x_1, \dots, x_p, x]\frac{1}{2^p}(T_{p-1}(x) - T_{p+1}(x)), \end{aligned} \tag{1.2}$$

where T_k and U_k are the Chebyshev polynomials of the first and second kinds, respectively. It is well known that

$$f[x_0, x_1, \dots, x_p, x] = \frac{f^{(p+1)}(\xi)}{(p+1)!},$$

for some $\xi \in (-1, 1)$. Now we introduce Condition M for the derivatives of f:

$$\max_{-1\le x\le 1} |f^{(k)}(x)| \le cM^k, \quad k = 0, 1, 2, \dots.$$

By the Stirling's formula

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n + \frac{1}{6}\right)}, \tag{1.3}$$

we have

$$|f(x) - f_I(x)| \le \frac{c}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p+1}. \tag{1.4}$$

This is a better convergent rate than $O(e^{-\sigma p})$ for any fixed $\sigma > 0$, since the right hand side of (1.4) is equivalent to

$$e^{p(\gamma - \ln p)}, \quad \gamma = \ln\frac{eM}{2}.$$

In the following sections, we shall realize the same convergence rate when the Chebyshev collocation method is used to solve two-point boundary value problems.

2 A Petrov–Galerkin Method

Consider the boundary value problem

$$-u'' = f, \qquad u(-1) = 0, \qquad u'(1) = 1. \tag{2.1}$$

Its weak form is to find $u \in H^1_{(0}[-1, 1] = \{v \in H^1[-1, 1] : v(-1) = 0\}$ such that

$$(u', v') = (f, v) + v(1), \quad \forall v \in H^1_{(0}[-1, 1]. \tag{2.2}$$

We introduce two families of auxiliary functions in $H^1_{(0}[-1,1]$:

$$\phi_1(x) = \pi - \arccos x, \qquad \phi_m(x) = \int_{-1}^{x} T_{m-1}(t)\frac{dt}{\sqrt{1-t^2}} = \frac{-\sqrt{1-x^2}\,U_{m-2}(x)}{m-1}, \quad m \geq 2, \tag{2.3}$$

$$\psi_m(x) = \int_{-1}^{x} T_{m-1}(t)dt, \quad m \geq 1. \tag{2.4}$$

Note that

$$\phi_1(-1) = 0 = \psi(-1), \qquad \phi_1(1) = \pi = (\psi_1', \phi_1'), \qquad \psi(1) = 2,$$
$$\phi_m(\pm 1) = 0 = \psi(\pm 1), \quad \text{for } m \geq 2,$$

$$(\psi_i', \phi_j') = \frac{\pi}{2}\delta_{ij}, \quad \text{except} \quad i = j = 1.$$

Our numerical approximation is the following.

(V0) Find $u_p \in \mathcal{P}_p[-1,1]$, $u_p(-1) = 0$, such that

$$(u_p', \phi_1') = (f, \phi_1) + \phi_1(1) = (f, \phi_1) + \pi,$$

$$(u_p', \phi_k') = (f, \phi_k), \quad m = 2, \ldots, p. \tag{2.5}$$

Based on the boundary condition, we have the expansion $u_p(x) = \sum_{k=1}^{p} b_{k-1}\psi_k(x)$. It is straightforward to obtain

$$b_0 = \frac{1}{\pi}(f, \phi_1) + 1. \tag{2.6}$$

$$b_{k-1} = \frac{2}{\pi}(f, \phi_k), \quad k = 2, \ldots, p. \tag{2.7}$$

As we will see in the proof of Theorem 2.1, u_p' is the truncation of the first p terms in the Chebyshev expansion for u'. Using the differential equation

$$(\sqrt{1-x^2}T_k'(x))' + \frac{k^2}{\sqrt{1-x^2}}T_k(x) = 0,$$

we can derive an expression for ψ_j:

$$\psi_{m+1}(x) = -\frac{1}{m^2-1}[(1-x^2)mU_{m-1}(x) + xT_m(x) + T_m(-1)]. \tag{2.8}$$

Remark The method is a Petrov–Galerkin type since we use different trial and test spaces. Furthermore, all basis functions in both spaces can be constructed by T_k and U_k, see (2.3) and (2.8).

Now we impose Condition M on u:

$$\max_{x\in[-1,1]} |u^{(k)}(x)| \leq cM^k, \quad k = 0, 1, 2, \ldots.$$

The relevance of Condition M was discussed in [18].

We need the following notation.

$$(u, v)_w = \int_{-1}^{1} \frac{u(x)v(x)dx}{\sqrt{1-x^2}}, \quad \|v\|_w = (v, v)_w^{1/2}.$$

Theorem 2.1 *Let u and u_p be solutions of (2.2) and (2.5), respectively. Assume that u satisfies Condition M and the non-degenerate condition: $|b_p| > |b_{p+1}|$. Then when $p+1 > M/\sqrt{2}$, we have*

$$(u - u_p)(x) \approx b_p \psi_{p+1}(x), \qquad (u - u_p)'(x) \approx b_p T_p(x), \tag{2.9}$$

$$\|u - u_p\|_w \approx O\left[\frac{1}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p+1}\right], \qquad \|u' - u'_p\|_w \approx O\left[\frac{1}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p}\right]. \tag{2.10}$$

Proof Actually, b_ms are the Chebyshev coefficients of u'. To see this, we start from the Chebyshev expansion

$$u'(x) = \sum_{n=0}^{\infty} c_n T_n(x),$$

and use the orthogonal property of T_js to verify, for $n \neq 0$,

$$c_n = \frac{2}{\pi} \int_{-1}^{1} \frac{u'(x)T_n(x)}{\sqrt{1-x^2}} dx = \frac{2}{\pi}(u', \phi'_{n+1}) = \frac{2}{\pi}(f, \phi_{n+1}) = b_n,$$

and for $n = 0$,

$$c_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{u'(x)T_0(x)}{\sqrt{1-x^2}} dx = \frac{1}{\pi}(u', \phi'_1) = \frac{1}{\pi}(f, \phi_1) + u'(1) = b_0.$$

Hence, the solution of (2.5) satisfies

$$u'_p(x) = \sum_{n=0}^{p-1} c_n T_n(x), \qquad (u - u_p)'(x) = \sum_{n \geq p} c_n T_n(x).$$

By Theorem 2.2.3 in [13, p. 71], we have

$$b_n = c_n = \frac{u^{(n)}(\xi_n)}{2^{n-1} n!}, \tag{2.11}$$

for a suitable $\xi_n \in (-1, 1)$. Therefore, by condition M,

$$\begin{aligned} \|u' - u'_p\|_w^2 &= \frac{\pi}{2} \sum_{n \geq p} c_n^2 \leq \frac{\pi}{2} \sum_{n \geq p} \left(\frac{cM^n}{2^{n-1} n!}\right)^2 \\ &= 2\pi c^2 \left(\frac{M^p}{2^p p!}\right)^2 \left(1 + \frac{M^2}{2^2(p+1)^2} + \frac{M^4}{2^4(p+1)^2(p+2)^2} + \cdots\right) \\ &\leq 4\pi c^2 \left(\frac{M^p}{2^p p!}\right)^2 \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right), \end{aligned}$$

when $p+1 > M/\sqrt{2}$. The last estimate in (2.10) follows by using the Stirling's formula. The first estimate in (2.10) can be obtained similarly from

$$u(x) = \sum_{n=0}^{\infty} b_n \psi_{n+1}(x)$$

by using (2.8).

Under the non-degenerate condition, $b_p\psi_{p+1}$ and b_pT_p are dominate terms in $u-u_p$ and $u'-u_p'$, respectively. □

Remark We refer the reader to [14] for another usage of the Petrov–Galerkin method in the spectral method.

3 Chebyshev Spectral Collocation

We consider collocation at $p+1$ Chebyshev points defined by (1.1). Here we seek $w_p \in \mathcal{P}_p \cap H^1_{(0}[-1,1]$ in the form

$$w_p(x) = a_0\psi_1(x) + a_1\psi_2(x) + \cdots + a_{p-1}\psi_p(x) \tag{3.1}$$

with $a_0 = b_0$, such that

$$-w_p''(x_j) = f(x_j), \quad x_j = \cos\frac{j\pi}{p}, \quad j = 1, \dots, p-1. \tag{3.2}$$

It is well known (see [4, p. 67] or [7, p. 104]) that when using

$$\omega_0 = \omega_p = \frac{\pi}{2p}, \qquad \omega_j = \frac{\pi}{p}, \quad j = 1, 2, \dots, p-1,$$

the related quadrature rule satisfies

$$\sum_{j=0}^{p} g(x_j)\omega_j = \int_{-1}^{1} g(x)\frac{dx}{\sqrt{1-x^2}}, \quad \forall g \in \mathcal{P}_{2p-1}.$$

Now we multiply both sides of (3.2) by $U_m(x_j)(1-x_j^2)\omega_j$ and sum up over $j = 1, 2, \dots, p-1$. Note that (1) it makes no difference with summing up over $j = 0, 1, \dots, p$ here; and (2) $w_p''(x)U_m(x)(1-x^2) \in \mathcal{P}_{2p-1}$ when $m \le p-1$ and consequently the integral on the left-hand side is exact. Therefore, for $m = 1, 2, \dots, p$,

$$\begin{aligned}
-&\sum_{j=1}^{p-1} w_p''(x_j)U_{m-1}(x_j)(1-x_j^2)\omega_j \\
&= -\int_{-1}^{1} w_p''(x)U_{m-1}(x)\sqrt{1-x^2}dx \\
&= \int_{-1}^{1} w_p'(x)(U_{m-1}(x)\sqrt{1-x^2})'dx
\end{aligned}$$

$$= -m \int_{-1}^{1} w_p'(x) T_m(x) \frac{dx}{\sqrt{1-x^2}}$$

$$= -m \int_{-1}^{1} w_p'(x) \phi_{m+1}'(x) dx = -m(w_p', \phi_{m+1}'). \tag{3.3}$$

On the other hand, the right-hand side is, for $m = 1, \dots, p-2$,

$$\sum_{j=1}^{p-1} f(x_j) U_{m-1}(x_j)(1-x_j^2)\omega_j$$

$$= \sum_{j=1}^{p-1} f_p(x_j) U_{m-1}(x_j)(1-x_j^2)\omega_j$$

$$= \int_{-1}^{1} f_I(x) U_{m-1}(x)\sqrt{1-x^2} dx = -m(f_I, \phi_{m+1}) \tag{3.4}$$

and

$$\sum_{j=1}^{p-1} f(x_j) U_{p-2}(x_j)(1-x_j^2)\omega_j = -(p-1)(f_I, \phi_p)^*, \tag{3.5}$$

where $(\cdot, \cdot)^*$ is the $(p+1)$-point Chebyshev–Lobatto quadrature rule defined above.

We see that the collocation method, is equivalent to the following variational formulation:
(V1) Find $w_p \in \mathcal{P}_p[-1, 1]$, $w_p(-1) = 0$, in the form (3.1) with $a_0 = b_0$, such that

$$(w_p', \phi_k') = (f_I, \phi_k)^*, \quad k = 2, \dots, p. \tag{3.6}$$

In light of (3.1),

$$w_p'(x) = a_0 + a_1 T_1(x) + \cdots + a_{p-1} T_{p-1}(x), \tag{3.7}$$

$$w_p''(x) = a_1 U_0(x) + 2a_2 U_1(x) + \cdots + (p-1)a_{p-1} U_{p-2}(x). \tag{3.8}$$

Substituting (3.7) into (3.6), we find, by (2.7), that

$$a_{p-1} = \frac{2}{\pi}(f_I, \phi_p)^*, \tag{3.9}$$

$$a_{k-1} = \frac{2}{\pi}(f_I, \phi_k) = b_{k-1}, \quad k = 2, 3, \dots, p-1. \tag{3.10}$$

Consider an auxiliary problem
(V2) Find $v_p \in \mathcal{P}_p[-1, 1]$, $v_p(-1) = 0$, in the form (3.1) with $a_0 = b_0$, such that

$$(v_p', \phi_k') = (f_I, \phi_k), \quad k = 2, \dots, p. \tag{3.11}$$

We can see that

$$(u_p' - v_p', \phi_k') = (f - f_I, \phi_k), \quad k = 2, \dots, p. \tag{3.12}$$

$$(v_p - w_p)(x) = (\bar{c}_{p-1} - a_{p-1})\psi_p(x), \quad \bar{c}_{p-1} = \frac{2}{\pi}(f_I, \phi_p), \tag{3.13}$$

where u_p and w_p are solutions of (V0) and (V1), respectively. Therefore,

$$\bar{c}_{p-1} - a_{p-1} = \frac{2}{\pi}[(f_I, \phi_p) - (f_I, \phi_p)^*]. \tag{3.14}$$

The numerical integration error can be estimated by [7, p. 104]

$$\int_{-1}^{1} g(x)\frac{dx}{\sqrt{1-x^2}} - \sum_{j=0}^{p} g(x_j)\omega_j = -\frac{\pi}{2^{2p-1}(2p)!}g^{(2p)}(\xi). \tag{3.15}$$

Hence,

$$(f_I, \phi_p) - (f_I, \phi_p)^* = -\frac{\pi}{2^{2p-1}(2p)!(p-1)}[f_I(x)U_{p-2}(x)(1-x^2)]^{(2p)}. \tag{3.16}$$

By the Newton–Leibniz formula,

$$\begin{aligned} &[f_I(x)U_{p-2}(x)(1-x^2)]^{(2p)} \\ &= \binom{2p}{p} f_I^{(p)}[U_{p-2}(1-x^2)]^{(p)} = \frac{1}{2}\frac{(2p)!}{p!^2} f_I^{(p)} T_p^{(p)} = f^{(p)}(\xi)\frac{2^{p-2}(2p)!}{p!}, \end{aligned} \tag{3.17}$$

for some $\xi \in (-1, 1)$. Here we have used the formula [11, p. 38]

$$2(1-x^2)U_{p-2}(x) = T_p(x) - T_{p-2}(x),$$

and the fact $T_p(x) = 2^{p-1}x^p + \cdots$. Substituting (3.17) into (3.16) and (3.16) into (3.14), we have

$$\bar{c}_{p-1} - a_{p-1} = \frac{f^{(p)}(\xi)}{2^p p!(p-2)}. \tag{3.18}$$

By (3.13), we have

$$(v_p - w_p)'(x) = (\bar{c}_{p-1} - a_{p-1})T_{p-1}(x) + h.o.t. \tag{3.19}$$

Therefore,

$$\|v_p' - w_p'\|_w = \sqrt{\frac{\pi}{2}}|\bar{c}_{p-1} - a_{p-1}| + h.o.t. \approx \frac{C}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p+1}. \tag{3.20}$$

On the other hand, with proper linear combination of ϕ_k in (3.12), we have

$$\begin{aligned} \int_{-1}^{1} (u_p - v_p)'(x)\frac{(u_p - v_p)'(x)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1} (f - f_I)(x)\int_{-1}^{x}\frac{(u_p - v_p)'(t)}{\sqrt{1-t^2}}dtdx \\ &= \int_{-1}^{1}\frac{(u_p - v_p)'(t)}{\sqrt{1-t^2}}\int_{t}^{1}(f - f_I)(x)dxdt. \end{aligned} \tag{3.21}$$

Therefore,

$$\|u_p' - v_p'\|_w^2 = (u_p' - v_p', E_p)_w, \quad E_p(t) = \int_t^1 (f - f_I)(s)ds.$$

Recall the interpolation error (1.4), we then have

$$\|u_p' - v_p'\|_w \leq \|E_p\|_w \approx \frac{C}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p+1}. \tag{3.22}$$

In light of Theorem 2.1, (3.20), and (3.22), we see that the difference $u' - u_p'$ is the dominant part for the error $u' - w_p'$ in the sense that

$$(u - w_p)'(x) \approx (u - u_p)'(x) = b_p T_p(x) + b_{p+1} T_{p+1}(x) + \cdots \approx \frac{C}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p}.$$

Here we have used (2.11), Condition M, and the Stirling's formula to estimate b_p. Let $y_j = (2j-1)\pi/(2p)$, $j = 1, \ldots, p$, i.e., zeros of T_p, then

$$(u - w_p)'(y_j) \approx (u - u_p)'(y_j) = b_{p+1} T_{p+1}(y_j) + \cdots \approx \frac{C}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p+1}.$$

Hence, The Chebyshev points y_j's are derivative superconvergent points. Results in this section can be summarized into the following.

Theorem 3.1 *Let u and w_p be solutions of* (2.2) *and* (3.2), *respectively. Assume that u satisfies Condition M and the non-degenerate condition*: $b_p \neq 0$. *Then when $p + 1 > M/\sqrt{2}$, we have*

$$(u - w_p)'(x) \approx b_p T_p(x), \qquad \|u' - w_p'\|_w \approx O\left[\frac{1}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p}\right]. \tag{3.23}$$

Furthermore, the zeros of T_p are superconvergent points for $u' - w_p'$ in the sense

$$(u - w_p)'(y_j) \approx O\left[\frac{1}{\sqrt{p}}\left(\frac{eM}{2p}\right)^{p+1}\right], \quad y_j = \frac{2j-1}{2p}\pi, \quad j = 1, 2, \ldots, p. \tag{3.24}$$

4 Further Extension

Consider the two-point boundary value problem

$$-u'' \pm \kappa^2 u = 0, \qquad u(-1) = 0, \qquad u'(1) = 1. \tag{4.1}$$

The solution satisfies Condition M with $M = \kappa$. We consider the case $f = 0$ for simplicity. For collocation methods, the only difference with $f \neq 0$ is in numerical integration, whose impact is of higher order as we have shown.

The weak form is to find $u \in H^1_{(0}[-1, 1]$ such that

$$(u', v') \pm \kappa^2 (u, v) = v(1), \quad \forall v \in H^1_{(0}[-1, 1].$$

The Chebyshev spectral collocation method of (4.1) is to find $w_p \in \mathcal{P}_p \cap H^1_{(0}[-1, 1]$ in the form of (3.1) such that

$$-w_p''(x_j) \pm \kappa^2 w_p(x_j) = 0, \quad x_j = \cos\frac{j\pi}{p}, \quad j = 1, \ldots, p-1, \tag{4.2}$$

and satisfying

$$a_0\left[1 \pm \frac{\kappa^2}{\pi}(\psi_1, \phi_1)\right] = 1 \mp \frac{\kappa^2}{\pi}\sum_{j=2}^{p} a_{j-1}(\psi_j, \phi_1). \tag{4.3}$$

The constraint (4.3) is from the Neumann boundary condition.

We can show that (4.2) is equivalent to the following:

$$(w_p', \phi_k') \pm \kappa^2 (w_p, \phi_k)^* = 0, \quad k = 2, \ldots, p. \tag{4.4}$$

Here $(\cdot, \cdot)^*$ is the same quadrature rule defined in Sect. 2. Let u_p be the Chebyshev expansion of the exact solution u of (4.1), it is straightforward to verify the following identity for $k = 2, \ldots, p$,

$$\begin{aligned}
&(w_p' - u_p', \phi_k') \pm \kappa^2 (w_p - u_p, \phi_k)^* \\
&\quad = (u' - u_p', \phi_k') \pm \kappa^2 (u - u_p, \phi_k)^* \pm \kappa^2[(u, \phi_k) - (u, \phi_k)^*] \\
&\quad = \pm\kappa^2 (u - u_p, \phi_k)^* \pm \kappa^2[(u, \phi_k) - (u, \phi_k)^*].
\end{aligned} \tag{4.5}$$

Let $u_I \in \mathcal{P}_p$ interpolate u at $p+1$ Chebyshev points x_j, $j = 0, 1, \ldots, p$, then

$$(u, \phi_k) - (u, \phi_k)^* = (u - u_I, \phi_k) + (u_I, \phi_k) - (u_I, \phi_k)^*.$$

Substituting this into (4.5), with some further analysis, we can show that the difference $w_p - u_p$ is of higher order comparing with $u - u_p$ under Condition M and the non-degenerate condition for sufficiently large p. As a consequence, $u - u_p$ is the dominant part of the error $u - w_p$ and similar error estimates as in Theorem 3.1 can be obtained for this case.

5 Final Remarks

1. The formulation and analysis for Dirichlet boundary conditions $u(-1) = 0 = u(1)$ are slightly simpler. In this case, we are seeking for $w_p \in \mathcal{P}_p \cap H_0^1[-1, 1]$ in the form

$$w_p(x) = a_1\psi_2(x) + \cdots + a_{p-1}\psi_p(x),$$

instead of (3.1). Note that $\psi_m(\pm 1) = 0$ for $m \geq 2$. The collocation is (4.2) without constraint (4.3). Then the analysis for the mixed boundary conditions is valid for the Dirichlet boundary conditions.

2. The similar results have been proved for the Legendre spectral collocation method [18]. Numerical tests indicate that the Chebyshev collocation is slightly better than the Legendre collocation under the maximum norm, even though they have the same convergence rate.

3. The super-geometric error estimate for the one-dimensional wave equation has been achieved for discrete dispersion relation for hp-version finite element approximation [1].

4. The extension to the two dimensional setting is not straightforward due to the corner singularities of polygonal domains. However, when the solutions are analytic as for some eigenvalue problems, the generalization of the one-dimensional results in this paper is feasible.

5. We interpret the collocation method at the Chebyshev points $x_j = \cos(j\pi/n)$ as a Chebyshev–Galerkin method which is almost equivalent to (up to some higher-order terms)

a Petrov–Galerkin method via the Chebyshev numerical integration (which is exact for polynomials of degree $\leq 2p-1$), rather than the Clenshaw–Curtis quadrature (which is exact for polynomials of degree $\leq p$), [6, 17].

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