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Numerical Solutions of Stochastic Control Problems for Regime-switching Systems

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Abstract. In this paper, we develop approximation methods for solving stochastic control problems. The systems under consideration involve regime switching, modulated by a continuous-time Markov chain. Using Markov chain approximation techniques, we construct discrete-time Markov chains having two components. In addition to convergence of the procedure, numerical experiments and remarks on controlled variance problems are presented.

Keywords. Stochastic control, regime-switching diffusion, Markov chain approximation.

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1 Introduction

Nowadays, many systems are complex and are of large scale, in which continuous dynamics and discrete events coexist. There are pressing needs of control design for such systems. Since the systems are often nonlinear, and since discrete events add further complexity, one frequently has to resort to numerical techniques to solve these problems. Much recent effort has been devoted to formulation, modeling, and optimization of regime-switching diffusions and regime-switching jump diffusions, which have attracted much attention in the past few years; see for example, [1, 3], among others. It has been demonstrated that such stochastic hybrid systems are more general and appropriate for a wide variety of applications. One of the distinctive features of the underlying system is that there are a finite number regimes across which the behavior of the system can be markedly different. For some recent applications in risk theory, financial engineering, and insurance modeling, see [2, 11, 12] and references therein. In fact, as argued in [8], the regime-switching models appear to be a good instrument to easily demonstrate such phenomena as volatility smile, which is otherwise difficult to observe. The regime-switching formulation has also been used in manufacturing, communication theory, signal processing, and wireless networks; see the many references cited in [7] and [10].

Similar to controlled diffusions, optimal controls of such hybrid systems yield a system of coupled Hamilton-Jacobi-Bellman (HJB) equations satisfied by the value functions. Even without regime switching, the HJB equations are usually nonlinear and difficult to solve in closed form. As a result, numerical methods become a viable alternative and possibly the only alternative in cases. In the literature, dealing with numerical methods for controlled diffusions and jump diffusions, one frequent choice of the numerical methods is the Markov chain approximation; see [4, 6]. Based on probabilistic methods, one constructs a Markov chain with specified transition probabilities leading to the approximation to the cost function, and the value functions etc.

In this work, we develop numerical algorithms for regime-switching controlled diffusions. We also use Markov chain approximation. Compared with the existing results in [6], in our problem, the systems are hybrid containing both continuous dynamics and discrete events. Thus the results of the aforementioned references are not directly applicable. We design numerical algorithms, and demonstrate their performance by considering some examples. Different from existing results on numerical methods for controlled diffusions, in lieu of one scalar cost function and one scalar value function, we have a collection of such functions. Thus effectively, we are dealing with a system instead of a single equation. Although we also use Markov chain approximation method, the approximating Markov chain has two components. One component is an approximation to the diffusion, whereas the other keeps track of the regimes.

The rest of the paper is arranged as follows. Formulation for controlled regime-switching diffusions is given in Section 2. In Section 3, we study the approximating Markov chain and its continuous-time interpolations. Also presented are relaxed control representations and convergence results. Several numerical examples are given in Section 4. Finally, Section 5 makes additional remarks on variance control problems.

2 Formulation

Consider a controlled hybrid diffusion system. For simplicity, assume that the system is one dimensional. The generalization to multidimensional systems is straightforward. Suppose that the discrete events take values in a finite set $\mathcal{M} = \{1, \ldots, m_0\}$ and $\alpha(\cdot)$ is a continuous-time Markov chain having state space $\mathcal{M}$ with generator $Q = (q_{\iota,\ell})$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{\mathcal{F}_t, t < \infty\}$ be a filtration defined on. Let $w(\cdot)$ be a standard $\{\mathcal{F}_t\}$-Wiener process. Let $u(\cdot)$ be a $\mathcal{F}_t$-adapted control, taking value in a compact set U. Such controls are called *admissible controls*. The dynamic system is:

$$dx(t) = b(x(t), \alpha(t), u(t))dt + \sigma(x(t), \alpha(t))dw(t), \quad x(0) = x, \alpha(0) = \iota, \tag{2.1}$$

where $x(t)$ is a component of the state representing the continuous dynamics and $\alpha(t)$ is another component of the state representing discrete events. For example, to model the price of a stock in a financial market, we use $\frac{dS}{S} =$

$\mu(\alpha(t))dt + \sigma(\alpha(t))dw$, where $S(\cdot)$ represent the stock price, $\mu(\cdot)$ and $\sigma(\cdot)$ are the appreciation and volatility rates, and $w(\cdot)$ is a standard Brownian motion. The use of the Markov chain $\alpha(\cdot)$ delineates the random environment (e.g., the market trends, as well as other economic factors).

To proceed, let τ be the first exit time from the interval $G = [0, B]$, i.e., $\tau = \min\{t : x(t) \notin G^o\}$, and consider the cost function

$$\begin{aligned} &W(x,\iota,u) = E^u_{x,\iota}\Big[\int_0^\tau \widetilde{k}(x(s),\alpha(s),u(s))ds + g(x(\tau),\alpha(\tau))\Big] \\ &W(x,\iota,u) = g(x,\iota), \quad \text{for} \quad x = 0, B \quad \text{and} \quad \iota \in \mathcal{M}, \end{aligned} \tag{2.2}$$

where $k(\cdot)$ and $g(\cdot)$ are appropriate functions representing the running cost and terminal cost, respectively. The notation $E^u_{x,\iota}$ means the expectation taken with the initial data $x(0) = x$ and $\alpha(0) = \iota$ and given control process $u(\cdot)$ used.

For an arbitrary $r \in U$, each $x \in G$, each $\iota \in \mathcal{M}$, and each $\phi(\cdot,\iota) \in C^2(\mathbb{R})$, define an operator L^r by

$$L^r\phi(x,\iota) = \phi_x(x,\iota)b(x,\iota,r) + \frac{1}{2}\phi_{xx}(x,\iota)\sigma^2(x,\iota) + Q\phi(x,\cdot)(\iota), \tag{2.3}$$

where $\phi_x(\cdot,\iota)$ and $\phi_{xx}(\cdot,\iota)$ denote the first and the second derivative with respect to x, and $Q\phi(x,\cdot)(\iota) = \sum_{\ell=1}^{m_0} q_{\iota\ell}\phi(x,\ell) = \sum_{\ell\neq\iota} q_{\iota\ell}(\phi(x,\ell) - \phi(x,\iota))$. Use $\mathcal{U}$ to denote the collection of all admissible controls. For each $\alpha \in \mathcal{M}$, let $V(x,\iota)$ be the value function

$$V(x,\iota) = \inf_{u\in\mathcal{U}} W(x,\iota,u), \tag{2.4}$$

Note that in lieu of one value function in the setup of controlled diffusions, we have a collection of value functions. They are solutions of the following Hamilton-Jacobi-Bellman (HJB) system of equations

$$\begin{aligned} &\inf_{r\in U}[L^rV(x,\iota) + \widetilde{k}(x,\iota,u)] = 0, \quad \iota \in \mathcal{M}, \\ &V(x,\iota) = g(x,\iota), \text{ for } x = 0, B. \end{aligned} \tag{2.5}$$

Our task to follow is to construct a numerical procedure for solving the optimal control problem. The method that we are using is Markov chain approximation; see [6]. However, our approximating Markov chain has two components. One of them represents the diffusive behavior and the other the switching process.

3 Discrete-time Approximating Markov Chain

Here, we construct a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion processes with regime switching. The approximating Markov chains will be *locally consistent* with (2.1) so that the weak limit of the Markov chain satisfies (2.1).

Construction of Markov Chain. Let $h > 0$ be a discretization step size. Define $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \dots\}$, and $\{(\xi^h_n, \alpha^h_n), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathcal{M}$ with transition probabilities from state $(x, \iota) \in \mathcal{M}$ to another state $(y, \ell) \in \mathcal{M}$, denoted by $p^h((x, \iota), (y, \ell)|r)$ for $r \in U$. We use u^h_n to denote the random variable that is the control action for the chain at discrete time n. In order to approximate the continuous time parameter process $(x(\cdot), \alpha(\cdot))$, we need to use an appropriate continuous-time interpolation. Suppose that we have an *interpolation interval* function $\Delta t^h(\cdot, \cdot, \cdot) > 0$ on $S_h \times \mathcal{M} \times U$, define $\Delta t^h_n = \Delta t^h(\xi^h_n, \alpha^h_n, u^h_n)$, and the interpolated time $t^h_n = \sum_{k=0}^{n-1} \Delta t^h_k(\xi^h_k, \alpha^h_k, u^h_k)$. Piecewise constant interpolations, denoted by $(\xi^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$ and $z^h(\cdot)$, are defined as

$$\xi^h(t) = \xi^h_n,\ \alpha^h(t) = \alpha^h_n,\ u^h(t) = u^h_n, z^h(t) = n, \quad \text{for } \ t \in [t^h_n, t^h_{n+1}). \tag{3.1}$$

As one of the necessary conditions for the weak convergence, we need the approximating Markov chains to be constructed satisfies *local consistency*.

Definition 3.1. Let $\{p^h((x, \iota), (y, \ell)|r)\}$ for $(x, \iota), (y, \ell) \in S^h \times \mathcal{M}$ and $r \in U$ be a collection of well defined transition probabilities for the two-component Markov chain (ξ^h_n, α^h_n), approximation to $(x(\cdot), \alpha(\cdot))$. Define the difference $\Delta\xi^h_n = \xi^h_{n+1} - \xi^h_n$. Assume $\inf_{x,\iota,r} \Delta t^h(x, \iota, r) > 0$ for each $h > 0$ and $\lim_{h\to 0} \sup_{x,\iota,r} \Delta t^h(x, \iota, r) \to 0$. Let $E^{r,h}_{x,\iota,n}$, $var^{r,h}_{x,\iota,n}$ and $p^{r,h}_{x,\iota,n}$ denote the conditional expectation, variance and probability given $\{\xi^h_k, \alpha^h_k, u^h_k, k \le n, \xi^h_n = x, \alpha^h_n = \iota, u^h_n = r\}$. The sequence (ξ^h_n, α^h_n) is said to be *locally consistent* with (2.1), if it satisfies

$$\begin{aligned}
&E^{r,h}_{x,\iota,n} \Delta\xi^h_n = b(x, \iota, r)\Delta t^h(x, \iota, r) + o(\Delta t^h(x, \iota, r)),\\
&var^{r,h}_{x,\iota,n} \Delta\xi^h_n = \sigma^2(x, \iota)\Delta t^h(x, \iota, r) + o(\Delta t^h(x, \iota, r)),\\
&p^{r,h}_{x,\iota,n}\{\alpha^h_{n+1} = \ell\} = \Delta t^h(x, \iota, r)q_{\iota\ell} + o(\Delta t^h(x, \iota, r)),\ \text{for } \ell \neq \iota,\\
&p^{r,h}_{x,\iota,n}\{\alpha^h_{n+1} = \iota\} = \Delta t^h(x, \iota, r)(1 + q_{\iota\iota}) + o(\Delta t^h(x, \iota, r)),\\
&\sup_{n,\omega\in\Omega} |\Delta\xi^h_n| \to 0 \ \text{ as } \ h \to 0.
\end{aligned} \tag{3.2}$$

With the approximating Markov chain given above, we can approximate the cost function defined in (2.2). Let $G^o_h = S_h \bigcap G^o$. Thus $G^o_h \times \mathcal{M}$ is the *finite state space* of the chain until it escapes from $G^o_h \times \mathcal{M}$. Denote the first time that $\{\xi^h_n\}$ leaves G^o_h by N_h. The cost functions for the chain that approximates (2.2) are

$$W^h(x, \iota, u^h) = E^{u^h}_{x,\iota} \sum_{n=0}^{N_h-1} \widetilde{k}(\xi^h_n, \alpha^h_n, u^h_n)\Delta t^h_n + E^{u^h}_{x,\iota} g(\xi^h_{N_h}, \alpha^h_{N_h}),\ \iota \in \mathcal{M}. \tag{3.3}$$

Corresponding to the continuous-time problems, the first term on the right-hand side of (3.3) represents the running cost and the last term gives the terminal cost. Using $\mathcal{U}^h$ to denote the collection of ordinary controls, which is determined by a conditional probability law

$$P\{u_n^h \in \cdot | \xi_k^h, \alpha_k^h, k \le n; u_k^h, k < n\}. \tag{3.4}$$

Theoretically, we can find approximation of $V(x,\iota)$ of (2.4) by using (3.3) and

$$V^h(x,\iota) = \inf_{u^h \in \mathcal{U}^h} W^h(x,\iota,u^h). \tag{3.5}$$

In actual computation, we compute $V^h(x,\iota)$ by solving the corresponding dynamic programming equation using iteration method. That is, for $\iota \in \mathcal{M}$,

$$V^h(x,\iota) = \begin{cases} \min_{r\in U}[\sum_{y,\ell} p^h((x,\iota),(y,\ell)|r)V^h(y,\ell) + \widetilde{k}(x,\iota,r)\Delta t^h(x,\iota,r)], \\ \hfill \text{for } \ x \in G_h^o, \\ g(x,\iota), \quad \text{for } \ x = 0, B. \end{cases} \tag{3.6}$$

Now we proceed to find the transition probabilities and interpolation interval functions for the Markov chain $\{(\xi_n^h, \alpha_n^h)\}$ to be constructed. To find a reasonable Markov chain that is locally consistent, we first suppose that control space has a unique admissible control $u^h \in \mathcal{U}^h$. In this case, min in (3.6) can be dropped. That is,

$$V^h(x,\iota) = \begin{cases} \sum_{y,\ell} p^h((x,\iota),(y,\ell)|r)V^h(y,\ell) + \widetilde{k}(x,\iota,r)\Delta t^h(x,\iota,r), \ x \in G_h^o, \\ g(x,\iota), \ x = 0, B. \end{cases} \tag{3.7}$$

Similarly, if we assume $\mathcal{U}$ has a unique admissible control $u(\cdot)$, we can drop inf in (2.5), and apply $r = u(0)$ for L^r. That is,

$$V_x(x,\iota)b(x,\iota,r) + \frac{1}{2}V_{xx}(x,\iota)\sigma^2(x,\iota) + \sum_\ell V(x,\ell)q_{\iota\ell} + \widetilde{k}(x,\iota,r) = 0 \tag{3.8}$$

We discretize (3.8) by following finite difference method using step-size $h > 0$,

$$\begin{aligned} &V(x,\iota) \to V^h(x,\iota) \\ &V_x(x,\iota) \to \frac{V^h(x+h,\iota) - V^h(x,\iota)}{h} \quad \text{for} \quad b(x,\iota,r) > 0, \\ &V_x(x,\iota) \to \frac{V^h(x,\iota) - V^h(x-h,\iota)}{h} \quad \text{for} \quad b(x,\iota,r) < 0, \\ &V_{xx}(x,\iota) \to \frac{V^h(x+h,\iota) - 2V^h(x,\iota) + V^h(x-h,\iota)}{h^2}. \end{aligned} \tag{3.9}$$

With boundary condition together, it leads to

$$
\begin{aligned}
&V^h(x,\iota) = g(x,\iota), \quad \text{for} \quad x = 0, B\\
&\frac{V^h(x+h,\iota) - V^h(x,\iota)}{h} b^+(x,\iota,r) - \frac{V^h(x,\iota) - V^h(x-h,\iota)}{h} b^-(x,\iota,r)\\
&\qquad + \frac{\sigma^2(x,\iota)}{2} \cdot \frac{V^h(x+h,\iota) - 2V^h(x,\iota) + V^h(x-h,\iota)}{h^2}\\
&\qquad + \sum_{\ell=1}^{m_0} q_{\iota\ell} V^h(x,\ell) + \widetilde{k}(x,\iota,r) = 0, \ \forall x \in G_h^o, \ \iota \in \mathcal{M},
\end{aligned}
\tag{3.10}
$$

where b^+ and b^- are the positive and negative parts of b, respectively. Combine the like terms, compare the result with (3.7), we obtain

$$
\begin{aligned}
&p^h((x,\iota),(x+h,\ell)|r) = \frac{(\sigma^2(x,\iota)/2) + hb^+(x,\iota,r)}{\widetilde{D}},\\
&p^h((x,\iota),(x-h,\iota)|r) = \frac{(\sigma^2(x,\iota)/2) + hb^-(x,\iota,r)}{\widetilde{D}},\\
&p^h((x,\iota),(x,\ell)|r) = \frac{h^2}{\widetilde{D}} q_{\iota\ell} \quad \text{for} \quad \ell \neq \iota,\\
&p^h(\cdot) = 0, \quad \text{otherwise}\\
&\Delta t^h(x,\iota,r) = \frac{h^2}{\widetilde{D}},
\end{aligned}
\tag{3.11}
$$

with $\widetilde{D} = \sigma^2(x,\iota) + h|b(x,\iota,r)| - h^2 q_{\iota\iota}$ being well defined. Next, we can show that the Markov Chain (ξ_n^h, α_n^h) with transition probabilities $\{p^h(\cdot)\}$ defined in (3.11) is locally consistent with (2.1).

Interpolations. Based on the Markov chain constructed, define the *continuous-time interpolation* $(\xi^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$ and $z^h(t)$ in (3.1). Use N_h defined above (3.3), define the first exit time of $\xi^h(\cdot)$ from G_h^o by $\tau_h = t_{N_h}^h$, and $\mathcal{D}_t^h$ as smallest σ-algebra of $\{\sigma^h(s), \alpha^h(s), u^h(s), z^h(s), s \leq t\}$. Then τ_h is $\mathcal{D}_t^h$-stopping time. Using the interpolation process, we can rewrite (3.3) as

$$
W^h(x,\iota,u^h) = E_{x,\iota}^{u^h}\Big[\int_0^{\tau_h} \widetilde{k}(\xi^h(s), \alpha^h(s), u^h(s))ds + g(\xi^h(\tau_h), \alpha^h(\tau_h))\Big]. \tag{3.12}
$$

In addition, $\mathcal{U}^h$ defined by (3.4) is equivalent to collection of all piecewise constant admissible controls with respect to $\mathcal{D}_t^h$. Hence, we still use the same formula for value function given in (3.5). To proceed, we need the following assumptions.

(H1) For each $\iota \in \mathcal{M}$ and each $r \in U$, the functions $b(\cdot,\iota,r)$ and $\sigma(\cdot,\iota)$ are continuous in G.

(H2) For each $\iota \in \mathcal{M}$, $\sigma(x,\iota) > 0, \forall x \in G$.

(H3) For each $\iota \in \mathcal{M}$ and each $r \in U$, the functions $\widetilde{k}(\cdot,\iota,r)$ and $g(\cdot,\iota)$ are continuous in G.

Use E_n^h to denote the conditional expectation given $\{\xi_k^h, \alpha_k^h, u_k^h, k \le n\}$. Define

$$M^h(t) = M_n^h,\ t \in [t_n^h, t_{n+1}^h), \quad \text{where} \quad M_n^h = \sum_{k=0}^{n-1} (\Delta\xi_k^h - E_k^h \Delta\xi_k^h). \tag{3.13}$$

Local consistency leads to

$$\begin{aligned} \xi^h(t) &= x + \sum_{k=0}^{z^h(t)-1} [E_k^h \Delta\xi_k^h + (\Delta\xi_k^h - E_k^h\Delta\xi_k^h)] \\ &= x + \sum_{k=0}^{z^h(t)-1} b(\xi_k^h, \alpha_k^h, u_k^h)\Delta t^h(\xi_k^h, \alpha_k^h, u_k^h) + \sum_{k=0}^{z^h(t)-1} (\Delta\xi_k^h - E_k^h\Delta\xi_k^h) \\ &\qquad + \varepsilon^h(t) \\ &= x + \int_0^t b(\xi^h(s), \alpha^h(s), u^h(s))dt + M^h(t) + \varepsilon^h(t). \end{aligned} \tag{3.14}$$

In the above, $\varepsilon^h(t)$ is the negligible error satisfying $\lim_{h\to 0}\sup_{s\le t} E|\varepsilon^h(t)| \to 0$. Note that $M^h(\cdot)$ is a martingale with respect to $\mathcal{D}_t^h$, and its discontinuities go to zero as $h \to 0$. Hence, we attempt to represent $M^h(t)$ in the form of similar to the diffusion term in (2.1). Define $w^h(\cdot)$ as

$$w^h(t) \stackrel{\text{def}}{=} \sum_{k=0}^{z^h(t)-1} (\Delta\xi_k^h - E_k^h\Delta\xi_k^h)/\sigma(\xi_k^h, \alpha_k^h). \tag{3.15}$$

Since $\sigma(\cdot) > 0$ in the compact set G, $\sigma^{-1}(\cdot)$ is uniformly bounded, which assures that the weak limit has continuous path with probability one. Note that in this paper, we are working with switching diffusions. The methods developed in what follows can be readily applied to switching systems of the form $(d/dt)x(t) = b(x(t), \alpha(t), u(t))$, without diffusion or $\sigma(\cdot,\cdot) \equiv 0$. Condition (H2) is a non-degeneracy requirement for the diffusion part, which is used for convenience for (3.15). This may be modified as $\sigma^\dagger(x,\iota) = \sigma^{-1}(x,\iota)$ if $\sigma(x,\iota) \ne 0$ and $\sigma^\dagger(x,\iota) = 0$ if $\sigma(x,\iota) = 0$, which is the trick used in general for martingale problems (see [6, p. 288]), but then more complex notation and another Brownian motion need to be introduced. We can now rewrite (3.14) as

$$\xi^h(t) = x + \int_0^t b(\xi^h(s), \alpha^h(s), u^h(s))ds + \int_0^t \sigma(\xi^h(s), \alpha^h(s))dw^h(s) + \varepsilon^h(t). \tag{3.16}$$

Relaxed Controls. To prove the desired convergence result, it is convenient to use the so-called relaxed controls. Its definition and properties can be found in [6, Section 4.6] for instance. Let $\mathcal{B}(\mathcal{U}\times[0,\infty))$ be the σ-algebra of Borel subsets of $\mathcal{U}\times[0,\infty)$. An *admissible relaxed control* or simply a *relaxed control* $m(\cdot)$ is a measure on $\mathcal{B}(\mathcal{U}\times[0,\infty))$ such that $m(\mathcal{U}\times[0,t])=t$ for all t. Given a relaxed control $m(\cdot)$, there is an $m_t(\cdot)$ such that $m(d\alpha dt)=m_t(d\alpha)dt$. In fact, we can define $m_t(A)=\lim_{\delta\to 0}\frac{m(A\times[t-\delta,t])}{\delta}$, $\forall A\in\mathcal{B}(U)$. where $\mathcal{B}(U)$ is the σ-algebra of Borel subsets of U. We can now define the relaxed control representation $m^h(\cdot)$ of $u^h(\cdot)$ by

$$m^h_t(A)=I_{\{u^h(t)\in A\}},\quad \forall A\in\mathcal{B}(U). \tag{3.17}$$

Let $\mathcal{F}^h_t$ denote the minimal σ-algebra that measures

$$\{\xi^h(s),\alpha^h(\cdot),m^h_s(\cdot),w^h(s),z^h(s),s\le t\}. \tag{3.18}$$

Apparently, $\mathcal{D}^h_t$ is smaller σ-algebra of $\mathcal{F}^h_t$. Using Γ^h to denote the set of all admissible control with respect to $\mathcal{F}^h_t$, and Γ^h is a larger control space containing $\mathcal{U}^h$. With the notion of relaxed control given above, we can write (3.16), (3.12), and the value function (3.5) as

$$\begin{aligned}\xi^h(t)=x\ &+\int_0^t\int_U b(\xi^h(s),\alpha^h(s),r)m^h_s(dr)ds\\ &+\int_0^t\sigma(\xi^h(s),\alpha^h(s))dw^h(s)+\varepsilon^h(t),\end{aligned} \tag{3.19}$$

$$\begin{aligned}W^h(x,\iota,m^h)\ &=E^{m^h}_{x,\iota}\int_0^{\tau^h}\int_U\widetilde{k}(\xi^h(s),\alpha^h(s),r)m^h_s(dr)ds\\ &+E^{m^h}_{x,\iota}g(\xi^h(\tau_h),\alpha^h(\tau_h)),\end{aligned} \tag{3.20}$$

and

$$V^h(x,\iota)=\inf_{m^h\in\Gamma^h}W^h(x,\iota,m^h). \tag{3.21}$$

The introduction of relaxed controls has the effect of making the control appear linearly in the dynamics and cost function. It turns out that the infimum of the cost over the classes of ordinary controls and relaxed controls are the same. We can rewrite (2.1) and (2.2) as

$$x(t)=x+\int_0^t\int_U b(x(s),\alpha(s),r)m_s(dr)ds+\int_0^t\sigma(x(s),\alpha(s))dw(s). \tag{3.22}$$

$$W(x,\iota,m)=E^m_{x,\iota}\Big[\int_0^\tau\int_U\widetilde{k}(x(s),\alpha(s),r)m_s(dr)ds+g(x(\tau),\alpha(\tau))\Big] \tag{3.23}$$

Now we are ready to give definition of existence and uniqueness of weak solution. We need two more assumptions.

(H4) Let $u(\cdot)$ be an admissible ordinary control with respect to $(w(\cdot),\alpha(\cdot))$, and suppose that $u(\cdot)$ is piecewise constant and takes only a finite number of values. Then for each initial condition, there exists a solution to (3.22) where $m(\cdot)$ is the relaxed control representation of $u(\cdot)$, and this solution is unique in the weak sense.

(H5) Let $\hat{\tau}(\phi) = \infty$, if $\phi(t) \in G^o$, for all $t < \infty$, otherwise, define $\hat{\tau}(\phi) = \inf\{t : \phi(t) \notin G^o\}$, then the function $\hat{\tau}(\cdot)$ is continuous (as a map from $D[0,\infty)$, the space of functions that are right continuous and have left limits endowed with the Skorohod topology, to the interval $[0,\infty]$ that is compactified) with probability one relative to the measure induced by any solution to (3.22) with initial condition (x,ι).

Convergence. Here we state a couple of convergence results. Their proofs can be found in [9]; we will not dwell on it.

Theorem 3.2. *Assume* (H1) *and* (H2). *Let* $\{\xi_n^h, \alpha_n^h, n < \infty\}$ *be constructed with transition probabilities defined in* (3.11), $\{u_n^h, n < \infty\}$ *be the sequence of admissible controls,* $(\xi^h(\cdot), \alpha^h(\cdot))$ *be the continuous-time interpolation defined in* (3.1), $m^h(\cdot)$ *be the relaxed control representation of* $\{u_n^h, n < \infty\}$, *and* $\{\tilde{\tau}_h\}$ *be a sequence of* $\mathcal{F}_t^h$*-stopping times. Then* $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot), \tilde{\tau}_h\}$ *is tight. Denote by* $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot), \tilde{\tau})$ *the limit of a weakly convergent subsequence and by* $\mathcal{F}_t$ *the* σ*-algebra generated by* $\{x(s), \alpha(s), m(s), w(s), s \le t, \tilde{\tau} I_{\{\tilde{\tau} \le t\}}\}$. *Then* $w(\cdot)$ *is a standard* $\mathcal{F}_t$*-Wiener process,* $\tilde{\tau}$ *is an* $\mathcal{F}_t$*-stopping time, and* $m(\cdot)$ *is an admissible control. Moreover, it satisfies (3.22).*

We next treat the convergence of the costs $W^h(x,\iota,m^h)$ given by (3.20), where $m^h(\cdot)$ is a sequence of admissible relaxed controls for $(\xi^h(\cdot), \alpha^h(\cdot))$. By virtue of Theorem 3.2, with the use of τ_h in (**??**) instead of $\tilde{\tau}_h$, we know that each sequence $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot), \tau_h\}$ of the type used in Theorem 3.2 has a weakly convergent subsequence whose limit process satisfies (3.22). With a slight abuse of notation, index the convergent subsequence by h with the limit denoted by $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot), \tilde{\tau})$. By assumption (H2), $\{\tau_h\}$ is uniformly integrable. By the weak convergence (Theorem 3.2) and the Skorohod representation, as $h \to 0$,

$$\begin{aligned} &E_{x,\iota}^{m^h} \int_0^{\tau^h} \int_U \widetilde{k}(\xi^h(s), \alpha^h(s), r) m_s^h(dr) ds \\ &\quad \to E_{x,\iota}^{m} \int_0^{\tilde{\tau}} \int_U \widetilde{k}(x(s), \alpha(s), r) m_s(dr) ds, \end{aligned} \tag{3.24}$$

$$E_{x,\iota}^{m^h} g(\xi^h(\tau_h), \alpha^h(\tau_h)) \to E_{x,\iota}^{m} g(x(\tilde{\tau}), \alpha(\tilde{\tau})). \tag{3.25}$$

Assumption (H5) guarantees that the first exit time of $x(\cdot)$ from G^o is $\tilde{\tau} = \tau$. This leads to

$$W^h(x,\iota,m^h) \to W(x,\iota,m) \quad \text{as} \quad h \to 0 \tag{3.26}$$

Theorem 3.3. *Assume* (H1)–(H5). $V^h(x,\iota)$ *and* $V(x,\iota)$ *are value functions defined in* (3.21) *and* (2.4), *respectively. Then* $V^h(x,\iota) \to V(x,\iota)$.

4 Numerical Examples

In this section, we provide several examples for demonstration. All numerical experiments were computed using MATLAB on a WinXP machine. The first example deals with a system that is linear in the state x as well as α, while the second example presents nonlinear dynamic system with respect to x and α. For numerical procedures, there are two commonly used methods, namely, approximation in policy space or policy iterations, and approximation in value space or value iterations. In [9], numerical experiments were done using iterations on policy space. Here, we use iteration on value space to demonstrate that such a procedure also works well.

Example 4.1 Let

$$dx(t) = (-3 + 2\alpha(t))(x(t) + u(t))dt + \frac{1}{2}\alpha(t)x(t)dw(t) \tag{4.1}$$

where the control $u(\cdot)$ takes value in a subset of $\mathbb{R}$, the Markov Chain $\alpha(\cdot) \in \mathcal{M}$ with $\mathcal{M} = \{1, 2\}$ and generator $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$, the set $G = [0, 2]$, and $U = [-2, 2]$. The cost function is given by

$$W(x, \iota, u) = E^u_{x,\iota} \int_0^\tau (x^2(t) + u^2(t))dt, \tag{4.2}$$

and the value function is $V(x, \iota) = \inf_{u(\cdot)} W(x, \iota, u)$.

Using the algorithms developed in this paper in conjunction with value iterations, we carry out the computations. For any $n \in \mathbb{Z}^+$, define vectors

$$\widetilde{V}^h_n = (V^h_n(h, 1), V^h_n(h, 2), \cdots, V^h_n(h, m_0), V^h_n(2h, 1), \cdots, V^h_n(|G^o_h|h, m_0))',$$

$$\widetilde{V}^h = (V^h(h, 1), V^h(h, 2), \cdots, V^h(h, m_0), V^h(2h, 1), \cdots, V^h(|G^o_h|h, m_0))'.$$

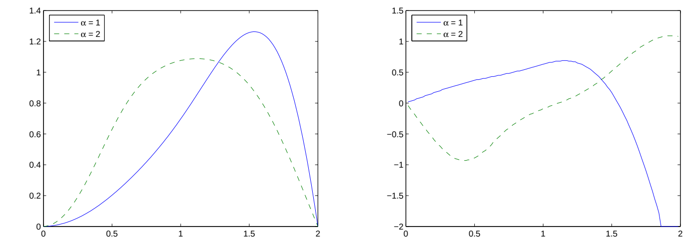
Using value iterations, we obtain $\widetilde{V}^h_n \to \widetilde{V}^h$ as $n \to \infty$,

1. Set $n = 0$. For all $x \in G^o_h$ and $\iota \in \mathcal{M}$, take initial value $V^h_0(x, \iota) = 1$;
2. find an improved values $\widetilde{V}^h_{n+1}(x, \iota)$,

$$V^h_{n+1}(x, \iota) = \min_{r \in U^h} [\sum_{y,\ell} p^h((x, \iota), (y, \ell)|r)V^h_n(y, \ell) + \widetilde{k}(x, \iota, r)\Delta t^h(x, \iota, r)],$$

$$u^h_{n+1}(x, \iota) = \text{argmin}_{r \in U^h} [\sum_{y,\ell} p^h((x, \iota), (y, \ell)|r)V^h_n(y, \ell) + \widetilde{k}(x, \iota, r)\Delta t^h(x, \iota, r)];$$

3. If $|\widetilde{V}^h_{n+1} - \widetilde{V}^h_n| >$ tolerance, then $n \to n + 1$ and go to step 2.



(a) Approximated Value Function (b) Optimal Feedback Control

Figure 1: Value Iteration with Step-size $h = 2^{-6}$ for Example 4.1

Figure 1 is the result of approximation of value and optimal control for each initial state in G_h^o with the step-size $h = 2^{-6}$. Table 1 presents the computation results for some initial states in comparison with various step-sizes. It demonstrates the convergence of the value. The iterative scheme was applied until the maximum difference between successive iterates was less than 10^{-3}.

Initial state	step size h			
	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$(0.5, 1)$	0.2262	0.2089	0.2030	0.2029
$(0.5, 2)$	0.5250	0.5332	0.5792	0.6299
$(1, 1)$	0.6250	0.6152	0.6330	0.6971
$(1, 2)$	0.7086	0.7608	0.9251	1.0760

Table 1: Values with Various Initial Points vs Step-size h for Example 4.1

It is difficult to analyze the rates of convergence theoretically. Nevertheless, for various step-size h, we obtain the error bounds computationally in Table 2.

Example 4.2 Consider a nonlinear model of (2.1).

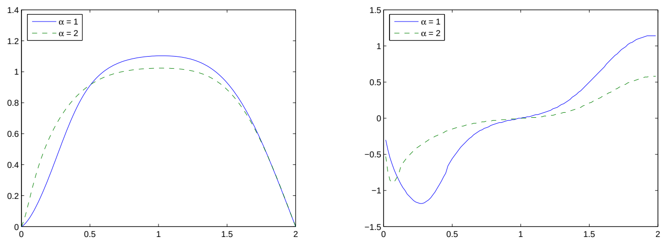
$$dx(t) = (x(t)^{1/2} + u(t)/\alpha(t))dt + \alpha(t)x(t)dw(t). \tag{4.3}$$

The rest parts are as in Example 4.1.

The value iteration algorithm is similar to that of Example 4.1. Computation results are in Figure 2 and Table 3. The error bounds are given in Table 4.

Initial state	step size h			
	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$(0.5, 1)$	0.2323 h	0.1871 h	0.1878 h	0.3668 h
$(0.5, 2)$	0.7324 h	1.3339 h	1.1944 h	0.8528 h
$(1, 1)$	0.3646 h	0.8868 h	1.2027 h	1.6982 h
$(1, 2)$	2.7077 h	4.5812 h	3.9054 h	1.8507 h

Table 2: Error Bounds with Various Initial Points vs Step-size h for Example 4.1



(a) Approximated Value Function (b) optimal feedback control function

Figure 2: Value Iteration with Step-size $h = 2^{-6}$ for Example 4.2

5 Remarks on Variance Control

So far, our effort has been devoted to numerical solutions for control problems when the variance is not controlled. In many applications, variance controls need to be dealt with. For example, in financial engineering problems as mentioned in the introduction, the variance is associated with the volatility. It is important to take variance control into consideration. However, even without regime switching, technical problems arise. As noted in [5], the basic problems for variance control are not in the proof of convergence, but in the actual construction of algorithms. Mainly, we need to design the algorithms such that they are convenient to code and do not introduce numerical noise. Numerical noise, also known as numerical viscosity, is numerical error introduced by the procedure that does not vanish as the discretization level goes to 0. Such errors may produce erroneous results in computation. It can be viewed as a lack of consistency. Roughly, with large numerical noise, the procedure will likely to converge to the optimal control of not the original

Initial	step size h			
state	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$(0.5, 1)$	0.6774	0.7428	0.8975	0.9108
$(0.5, 2)$	0.4011	0.5091	0.7714	0.9145
$(1, 1)$	0.7199	0.7880	1.0203	1.1029
$(1, 2)$	0.3970	0.5017	0.8364	1.0233

Table 3: Values with Various Initial Points vs Step-size h for Example 4.2

Initial	step size h			
state	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$(0.5, 1)$	2.0922 h	3.1380 h	1.3265 h	1.8014 h
$(0.5, 2)$	4.1052 h	6.4814 h	4.5702 h	0.0202 h
$(1, 1)$	3.1328 h	5.1757 h	2.9179 h	0.5521 h
$(1, 2)$	4.9127 h	8.1505 h	5.5922 h	0.7804 h

Table 4: Error Bounds with Various Initial Points vs Step-size h for Example 4.2

system, but a system with larger variance. Thus it is of importance to reduce these numerical noise.

Different from the system given by (2.1), nonlocal transitions that are transitions to states other than nearest neighbor could be unavoidable. We illustrate the situation by considering the following simple example:

$$dx = \sigma(\alpha(t))xudw.$$

For simplicity, we only consider the controlled diffusion part. In fact, the drift part can be treated separately. Suppose that $\alpha(t)$ has two states $\{1, 2\}$. This stems from the motivation of the description of a stock price, where $\alpha(t)$ represents the bullish or bearish states of the random environment. Let δ be the discretization interval. Consider the approximation of $x \to x + \sigma(i)xr\beta$, where β is a random variable with mean zero and variance h. It follows that randomization between grid points is needed. We can develop an explicit scheme similar to [5, p. 2290]. We could also develop implicit schemes. There are quite few number of subtle issues. To consider them and to treat these problems carefully under regime switching deserve further study and exploration.

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