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Numerical methods for controlled regime-switching diffusions and regime-switching jump diffusions☆

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Abstract

This work is concerned with numerical methods for controlled regime-switching diffusions, and regime-switching jump diffusions. Numerical procedures based on Markov chain approximation techniques are developed. Convergence of the algorithms is derived by means of weak convergence methods. In addition, examples are also provided for demonstration purpose.

Keywords: Controlled regime-switching diffusion; Regime-switching jump diffusion; Markov chain approximation; Convergence

1. Introduction

Many systems in the real world are complex, in which continuous dynamics and discrete events coexist. The need of successfully control such systems in practice leads to the resurgent effort in formulation, modeling, and optimization of regime-switching diffusions and regime-switching jump diffusions. It has attracted much needed attention in the last few years; see for example, Blair and Sworder (1986), Ji and Chizeck (1990), Mariton and Bertrand (1985); Mao (1999), among others. Recent study of stochastic hybrid systems has indicated that such a formulation is more general and appropriate for a wide variety of applications. One of the distinctive features of the underlying system is that there are a number of regimes across which the behavior of the system can be markedly different. For some recent applications in risk theory, financial engineering, and insurance modeling, we refer the reader to Di Masi, Kabanov, and Runggaldier (1994), Dufresne and Gerber (1991), Moller (1995), Rolski, Schmidli, Schmidt, and Teugels (1999), Yang and Yin (2004), Yin, Liu, and Zhang (2002), Zhang (2001) and references therein. Such a formulation has also been used in manufacturing, communication theory, signal processing, and wireless networks; see the many references cited in Kushner and Yin (2003), Yin and Zhang (2005). Loosely, the state of the system consists of two components. One of them describes the continuous dynamics, and the other models discrete events. The discrete event is modeled by a Markov chain representing the possible regimes, whereas the continuous dynamics are diffusion processes. It is well known that optimal controls of such systems lead to systems of Hamilton–Jacobi–Bellman (HJB) equations satisfied by the value functions. Even without regime switching, the HJB equations are usually nonlinear and difficult to solve in closed form. Thus numerical methods become viable alternative. One of the most effective methods is the Markov chain approximation approach; see Kushner (1990), Kushner and Dupuis (2001). Based on probabilistic methods, one constructs a Markov chain with specified transition probabilities leading to the approximation to the cost function, and the value functions etc. [Related numerical methods for solving stochastic differential equations can be found in for example, Kloeden and Platen, 1992; Milstein, 1995; Platen, 1999, Protter and Talay, 1997, among others]. Although regime-switching diffusions are important for many applications, the numerical methods for optimal controls of such systems are still scarce.

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In this work, we develop numerical algorithms for regime-switching controlled diffusions and regime-switching jump diffusions, prove their convergence, and demonstrate their performance by considering some examples. Different from existing results on numerical methods for controlled diffusions, in lieu of one scalar cost function and one scalar value function, we have a collection of such functions. Effectively, we are dealing with a system instead of a single equation. Although we also use Markov chain approximation method, compared with the existing results in Kushner and Dupuis (2001), the systems are hybrid containing both continuous dynamics and discrete events. The results of the aforementioned references are not directly applicable. In our problem, the approximating Markov chain has two components. One component is an approximation to the diffusion, whereas the other keeps track of the regimes.

The rest of the paper is arranged as follows. Problem formulation for controlled regime-switching diffusions is given next. In Section 3, we study the approximating Markov chain, and in Section 4, we consider interpolated processes of the approximation. In Section 5, relaxed control representation is introduced for our approximation. Section 6 establishes the convergence of the algorithms. Section 7 extends the formulation and results to regime-switching jump diffusions and discounted cost problems. Several numerical examples are given in Section 8. Section 9 makes additional remarks. Finally, an appendix is provided to include the detailed proofs of results.

2. Formulation

Consider a controlled hybrid diffusion system or controlled diffusions with regime switching. For simplicity, the system is assumed to be one dimensional; it can be easily generalized to multi-dimensional cases. Suppose that there is a finite set $\mathscr{M}=\{1,\ldots,m_0\}$ representing the possible regimes of the environment, that $\alpha(\cdot)$ is a continuous-time Markov chain having state space $\mathscr{M}$ with generator $Q=(q_{\imath\ell})$, and that $w(\cdot)$ is a standard Wiener process. Let $\{\mathscr{F}_t\}$ be a filtration that measures at least $\{w(s),\alpha(s) : s\leqslant t\}$, and $u(\cdot)$ be an $\mathscr{F}_t$-adapted control, taking value in a compact set $U\subset\mathbb{R}$. Such controls are said to be *admissible controls*. The dynamic system of interest is

$$\begin{aligned}&\mathrm{d}x(t)=b(x(t),\alpha(t),u(t))\,\mathrm{d}t+\sigma(x(t),\alpha(t))\,\mathrm{d}w(t),\\&x(0)=x,\quad \alpha(0)=\imath,\end{aligned}\tag{1}$$

where $x(t)$ is a component of the state representing the continuous dynamics and $\alpha(t)$ is another component representing discrete events. For example, to model the price of a stock in a financial market, we use $\mathrm{d}S=\mu(\alpha(t))S\,\mathrm{d}t+\sigma(\alpha(t))S\,\mathrm{d}w$,where $S(\cdot)$ represent the stock price, $\mu(\cdot)$ and $\sigma(\cdot)$ the appreciation and volatility rates, and $w(\cdot)$ a standard Brownian motion. The use of the Markov chain $\alpha(\cdot)$ is an effort to represent the random environment, the market trends, as well as other economic factors.

To proceed, let τ be the first exit time from the interval $G=[0,B]$, i.e.,

$$\tau=\min\{t : x(t)\notin G^0\},\tag{2}$$

and consider the cost function

$$\begin{aligned}W(x,\imath,u)&=E^u_{x,\imath}\left[\int_0^\tau \widetilde{k}(x(s),\alpha(s),u(s)\,\mathrm{d}s\right.\\&\qquad\left.+\,g(x(\tau),\alpha(\tau))\right],\end{aligned}\tag{3}$$

$$W(x,\imath,u)=g(x,\imath)\quad\text{for }(x,\imath)\in\{0,B\}\times\mathscr{M},$$

where $\widetilde{k}(\cdot)$ and $g(\cdot)$ are appropriate functions representing the running cost and terminal cost, respectively. In the above, the notation $E^u_{x,\imath}$ denotes the expectation taken with the initial data $x(0)=x$ and $\alpha(0)=\imath$ and given control process $u(\cdot)$ used. For an arbitrary $r\in U$, $(x,\imath)\in G\times\mathscr{M}$, and $\phi(\cdot,\imath)\in C^2(\mathbb{R})$, define an operator L^r by $L^r\phi(x,\imath)=\phi_x(x,\imath)b(x,\imath,r)+\frac{1}{2}\phi_{xx}(x,\imath)\sigma^2(x,\imath)+Q\phi(x,\cdot)(\imath)$, where $\phi_x(\cdot,\imath)$ and $\phi_{xx}(\cdot,\imath)$ denote the first and second derivative with respect to x, and $Q\phi(x,\cdot)(\imath)=\sum_{\ell=1}^{m_0}q_{\imath\ell}\phi(x,\ell)=\sum_{\ell\neq\imath}q_{\imath\ell}(\phi(x,\ell)-\phi(x,\imath))$. Let $\mathscr{U}$ be the collection of admissible controls. For each $\imath\in\mathscr{M}$, let $V(x,\imath)$ be the value function

$$V(x,\imath)=\inf_{u\in\mathscr{U}}W(x,\imath,u).\tag{4}$$

The value functions are solutions of the following system of HJB equations

$$\begin{aligned}&\inf_{r\in U}[L^rV(x,\imath)+\widetilde{k}(x,\imath,r)]=0,\ \ (x,\imath)\in G^0\times\mathscr{M},\\&V(x,\imath)=g(x,\imath)\quad\text{for }(x,\imath)\in\{0,B\}\times\mathscr{M}.\end{aligned}\tag{5}$$

Our task to follow is to construct a numerical procedure for solving the optimal control problem. The method that we are using is Markov chain approximation; see Kushner and Dupuis (2001). However, our approximating Markov chain has two components. One of them delineates the behavior of the continuous component and the other represents the switching process.

3. Approximating Markov chain

In this section, we construct a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion processes with regime switching. The approximating Markov chain is *locally consistent* with (1), so that the weak limit of the Markov chain satisfies (1). Let $h>0$ be a discretization parameter. Define $S_h=\{x : x=kh, k=0,\pm1,\pm2,\ldots\}$. Let $\{(\xi^h_n,\alpha^h_n), n<\infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h\times\mathscr{M}$ with transition probabilities from a state $(x,\imath)\in\mathscr{M}$ to another state $(y,\ell)\in\mathscr{M}$, denoted by $p^h((x,\imath),(y,\ell)|r)$ for $r\in U$. We use u^h_n to denote the random variable that is the control action for the chain at discrete time n. In order to approximate the continuous-time $(x(\cdot),\alpha(\cdot))$, we need to use an appropriate continuous-time interpolation (Kushner & Dupuis, 2001). [Note that due to the addition of the Markov chain $\alpha(t)$, the interpolations we are taking is a generalization of the aforementioned reference.] Suppose that we have an *interpolation interval* $\Delta t^h(\cdot,\cdot,\cdot)>0$ on $S_h\times\mathscr{M}\times U$, and denote $\Delta t^h_n=\Delta t^h(\xi^h_n,\alpha^h_n,u^h_n)$. Define the interpolated time $t^h_n=\sum_{k=0}^{n-1}\Delta t^h_k(\xi^h_k,\alpha^h_k,u^h_k)$. Hence, the piecewise constant interpolations, denoted by $(\xi^h(\cdot),\alpha^h(\cdot))$, $u^h(\cdot)$ and $z^h(\cdot)$, are naturally defined as, for $t \in [t_n^h, t_{n+1}^h)$,

$$\xi^h(t) = \xi_n^h, \quad \alpha^h(t) = \alpha_n^h, \quad u^h(t) = u_n^h, \quad z^h(t) = n. \tag{6}$$

We need the approximating Markov chain constructed satisfying *local consistency*.

Definition 1. Let $\{p^h((x, \imath), (y, \ell)|r)\}$ for $(x, \imath), (y, \ell) \in S^h \times \mathscr{M}$ and $r \in U$ be a collection of well defined transition probabilities for the two-component Markov chain (ξ_n^h, α_n^h), approximation to $(x(\cdot), \alpha(\cdot))$. Define the difference $\Delta\xi_n^h = \xi_{n+1}^h - \xi_n^h$. Assume $\inf_{x,\imath,r}\Delta t^h(x, \imath, r) > 0$ for each $h > 0$ and $\lim_{h\to 0}\sup_{x,\imath,r}\Delta t^h(x, \imath, r) \to 0$. Let $E_{x,\imath,n}^{r,h}$, $var_{x,\imath,n}^{r,h}$ and $p_{x,\imath,n}^{r,h}$ denote the conditional expectation, variance, and marginal probability given $\{\xi_k^h, \alpha_k^h, u_k^h, k \leqslant n, \xi_n^h = x, \alpha_n^h = \imath, u_n^h = r\}$, respectively. The sequence $\{(\xi_n^h, \alpha_n^h)\}$ is said to be *locally consistent* with (1), if it satisfies, for $\varepsilon^h = o(\Delta t^h(x, \imath, r))$,

$$\begin{aligned}
&E_{x,\imath,n}^{r,h}\Delta\xi_n^h = b(x, \imath, r)\Delta t^h(x, \imath, r) + \varepsilon^h,\\
&var_{x,\imath,n}^{r,h}\Delta\xi_n^h = \sigma^2(x, \imath)\Delta t^h(x, \imath, r) + \varepsilon^h,\\
&p_{x,\imath,n}^{r,h}\{\alpha_{n+1}^h = \ell\} = \Delta t^h(x, \imath, r)q_{\imath\ell} + \varepsilon^h \quad \text{for } \ell \neq \imath,\\
&p_{x,\imath,n}^{r,h}\{\alpha_{n+1}^h = \imath\} = \Delta t^h(x, \imath, r)(1 + q_{\imath\imath}) + \varepsilon^h,\\
&\sup_{n,\omega\in\Omega} |\Delta\xi_n^h| \to 0 \text{ as } h \to 0.
\end{aligned} \tag{7}$$

Suppose we have the approximating Markov chain discussed above. Then we can obtain approximation of cost function defined in (3). Let $G_h^0 = S_h \cap G^0$. Thus $G_h^0 \times \mathscr{M}$ is a finite state space. Let N_h denote the first time that $\{\xi_n^h\}$ leaves G_h^0. Natural cost functions for the chain that approximates (3) are, for $(x, \imath) \in G_h^0 \times \mathscr{M}$,

$$\begin{aligned}
W^h(x, \imath, u^h) = {} & E_{x,\imath}^{u^h} \sum_{n=0}^{N_h-1} \widetilde{k}(\xi_n^h, \alpha_n^h, u_n^h)\Delta t_n^h\\
& + E_{x,\imath}^{u^h} g(\xi_{N_h}^h, \alpha_{N_h}^h).
\end{aligned} \tag{8}$$

Corresponding to the continuous-time problems, the first term on the right-hand side of (8) represents the running cost and the last term gives the terminal cost. Using $\mathscr{U}^h$ to denote the collection of controls, which are determined by a sequence of measurable functions $F_n^h(\cdot)$ such that $u_n^h = F_n^h(\xi_k^h, \alpha_k^h, k \leqslant n; u_k^h, k < n)$. Theoretically, we can find approximation of $V(x, \imath)$ of (4) by

$$V^h(x, \imath) = \inf_{u^h\in\mathscr{U}^h} W^h(x, \imath, u^h). \tag{9}$$

Practically, we can compute $V^h(x, \imath)$ by solving the corresponding dynamic programming equation using iteration method. That is, for $(x, \imath) \in G_h^0 \times \mathscr{M}$,

$$\begin{aligned}
V^h(x, \imath) = \min_{r\in U} \Bigg\{ & \sum_{y,\ell} p^h((x, \imath), (y, \ell)|r)V^h(y, \ell)\\
& + \widetilde{k}(x, \imath, r)\Delta t^h(x, \imath, r) \Bigg\},
\end{aligned} \tag{10}$$

with the boundary condition $V^h(x, \imath) = g(x, \imath)$ for $(x, \imath) \in \{0, B\} \times \mathscr{M}$.

Now we proceed to find the transition probabilities and interpolation intervals for the Markov chain $\{(\xi_n^h, \alpha_n^h)\}$. To find a reasonable Markov chain that is locally consistent, we first consider a special case, in which the control space has a unique admissible control $u^h \in \mathscr{U}^h$. In this case, min in (10) can be dropped. That is,

$$\begin{aligned}
V^h(x, \imath) = {} & \sum_{y,\ell} p^h((x, \imath), (y, \ell)|r)V^h(y, \ell)\\
& + \widetilde{k}(x, \imath, r)\Delta t^h(x, \imath, r).
\end{aligned} \tag{11}$$

Similarly, if we assume $\mathscr{U}$ has a unique admissible control $u(\cdot)$, we can drop inf in (5), and apply $r = u(0)$ in L^r. That is,

$$\begin{aligned}
& V_x(x, \imath)b(x, \imath, r) + \frac{1}{2}V_{xx}(x, \imath)\sigma^2(x, \imath)\\
& \quad + \sum_{\ell} V(x, \ell)q_{\imath\ell} + \widetilde{k}(x, \imath, r) = 0.
\end{aligned} \tag{12}$$

Discretize (12) using upwind finite difference method with stepsize $h > 0$ by

$$\begin{aligned}
& V(x, \imath) \to V^h(x, \imath)\\
& V_x(x, \imath) \to \frac{V^h(x + h, \imath) - V^h(x, \imath)}{h} \quad \text{for } b(x, \imath, r) > 0,\\
& V_x(x, \imath) \to \frac{V^h(x, \imath) - V^h(x - h, \imath)}{h} \quad \text{for } b(x, \imath, r) < 0,\\
& V_{xx}(x, \imath) \to \frac{V^h(x + h, \imath) - 2V^h(x, \imath) + V^h(x - h, \imath)}{h^2}.
\end{aligned}$$

For $(x, \imath) \in G_h^0 \times \mathscr{M}$, it leads to

$$\begin{aligned}
& \frac{V^h(x + h, \imath) - V^h(x, \imath)}{h} b^+(x, \imath, r)\\
& \quad - \frac{V^h(x, \imath) - V^h(x - h, \imath)}{h} b^-(x, \imath, r)\\
& \quad + \frac{\sigma^2(x, \imath)}{2} \cdot \frac{V^h(x + h, \imath) - 2V^h(x, \imath) + V^h(x - h, \imath)}{h^2}\\
& \quad + \sum_{\ell=1}^{m_0} q_{\imath\ell}V^h(x, \ell) + \widetilde{k}(x, \imath, r) = 0,
\end{aligned}$$

where b^+ and b^- are the positive and negative parts of b, respectively. Combining like terms and comparing the result with (11), we obtain transition probability

$$\begin{aligned}
& p^h((x, \imath), (x + h, \ell)|r) = \frac{(\sigma^2(x, \imath)/2) + hb^+(x, \imath, r)}{\widetilde{D}},\\
& p^h((x, \imath), (x - h, \imath)|r) = \frac{(\sigma^2(x, \imath)/2) + hb^-(x, \imath, r)}{\widetilde{D}},\\
& p^h((x, \imath), (x, \ell)|r) = \frac{h^2}{\widetilde{D}}q_{\imath\ell} \quad \text{for } \ell \neq \imath,\\
& p^h(\cdot) = 0 \quad \text{otherwise},\\
& \Delta t^h(x, \imath, r) = \frac{h^2}{\widetilde{D}},
\end{aligned} \tag{13}$$

with $\widetilde{D} = \sigma^2(x, \imath) + h|b(x, \imath, r)| - h^2 q_{\imath\imath}$ being well defined. Next, we present the local consistency for our approximation sequence. The proof of the following lemma is to verify all conditions in Definition 1 through calculations, and is omitted.

Lemma 2. *The Markov chain* (ξ_n^h, α_n^h) *with transition probabilities* $\{p^h(\cdot)\}$ *defined in* (13) *is locally consistent with* (1).

4. Interpolations

Based on the Markov chain approximation constructed in the last section, piecewise constant interpolation is obtained here with appropriately chosen interpolation intervals. Using $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ to approximate the continuous-time process $(x(\cdot), \alpha(\cdot))$, we defined the *continuous-time interpolation* $(\xi^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$ and $z^h(t)$ in (6). Using N_h given above (8), define the first exit time of $\xi^h(\cdot)$ from G_h^0 by $\tau_h = t_{N_h}^h$. Denote the σ-algebra of $\{\sigma^h(s), \alpha^h(s), u^h(s), z^h(s), s \leqslant t\}$ by $\mathscr{D}_t^h$. Then τ_h is a $\mathscr{D}_t^h$-stopping time. Using the interpolation process, we can rewrite (8) as

$$W^h(x, \imath, u^h) = E_{x,\imath}^{u^h}\left[\int_0^{\tau_h} \widetilde{k}(\xi^h(s), \alpha^h(s), u^h(s))\,\mathrm{d}s + g(\xi^h(\tau_h), \alpha^h(\tau_h))\right]. \tag{14}$$

In addition, $\mathscr{U}^h$ is equivalent to the collection of all piecewise constant admissible controls with respect to $\mathscr{D}_t^h$. Hence, we still use the same formula for value function given in (9). To proceed, we need the following assumptions:

(A1) For each $\imath \in \mathscr{M}$ and each $r \in U$, the functions $b(\cdot, \imath, r)$ and $\sigma(\cdot, \imath)$ are continuous in G.
(A2) For each $\imath \in \mathscr{M}$, $\sigma(x, \imath) > 0, \forall x \in G$.
(A3) For each $\imath \in \mathscr{M}$ and each $r \in U$, the functions $\widetilde{k}(\cdot, \imath, r)$ and $g(\cdot, \imath)$ are continuous in G.

Use E_n^h to denote the conditional expectation given $\{\xi_k^h, \alpha_k^h, u_k^h, k \leqslant n\}$. Define $M^h(t) = M_n^h,\ t \in [t_n^h, t_{n+1}^h)$, where $M_n^h = \sum_{k=0}^{n-1}(\Delta\xi_k^h - E_k^h\Delta\xi_k^h)$. The local consistency leads to

$$\begin{aligned}\xi^h(t) &= x + \sum_{k=0}^{z^h(t)-1}[E_k^h\Delta\xi_k^h + (\Delta\xi_k^h - E_k^h\Delta\xi_k^h)] \\ &= x + \int_0^t b(\xi^h(s), \alpha^h(s), u^h(s))\,\mathrm{d}t + M^h(t) + \varepsilon^h(t),\end{aligned} \tag{15}$$

with error $e^h(t)$ satisfying $\lim_{h\to 0}\sup_{0\leqslant t\leqslant T} E|\varepsilon^h(t)| \to 0$ for any $0 < T < \infty$. Note that $M^h(\cdot)$ is a martingale with respect to $\mathscr{D}_t^h$, and its discontinuities go to zero as $h \to 0$. We attempt to represent $M^h(t)$ similar to the diffusion term in (1). Define $w^h(\cdot)$ as

$$\begin{aligned}w^h(t) &\stackrel{\text{def}}{=} \sum_{k=0}^{z^h(t)-1}(\Delta\xi_k^h - E_k^h\Delta\xi_k^h)/\sigma(\xi_k^h, \alpha_k^h), \\ &= \int_0^t \sigma^{-1}(\xi^h(s), \alpha^h(s))dM^h(s).\end{aligned} \tag{16}$$

We can now rewrite (15) as

$$\xi^h(t) = x + \int_0^t b(\xi^h(s), \alpha^h(s), u^h(s))\,\mathrm{d}s + \int_0^t \sigma(\xi^h(s), \alpha^h(s))\,\mathrm{d}w^h(s) + \varepsilon^h(t). \tag{17}$$

Since $\sigma(\cdot) > 0$ in the compact set G, $\sigma^{-1}(\cdot)$ is uniformly bounded, which assures that the weak limit has continuous path with probability one.

Note that in this paper, we are working with switching diffusions. The methods developed in what follows can be readily applied to switching systems of the form $(\mathrm{d}/\mathrm{d}t)x(t) = b(x(t), \alpha(t), u(t))$, without diffusion or $\sigma(\cdot,\cdot) \equiv 0$. Condition (A2) is a non-degeneracy requirement for the diffusion part, which is used for convenience. In case, if σ is not strictly positive, we modify its inverse by $\sigma^\dagger(x, \imath) = \sigma^{-1}(x, \imath)$ if $\sigma(x, \imath) \neq 0$ and $\sigma^\dagger(x, \imath) = 0$ if $\sigma(x, \imath) = 0$, which is a trick used in general for martingale problems (see Kushner & Dupuis, 2001, p. 288), and which requires more complex notation and the use of another Brownian motion.

5. Relaxed control

Sections 3 and 4 gave numerical method to approximate $V(\cdot)$ in (3). Only weak sense solution of (1) is important. Our primary goal is to prove convergence of our approximation to desired $V(\cdot)$ as $h \to 0$. The sequence of ordinary controls might not converge in a traditional sense, and the use of the relaxed control terminology enables us to obtain and appropriately characterize the weak limit. To facilitate the proof of weak convergence, we introduce the relaxed control representation; see Kushner and Dupuis (2001, Section 4.6) for details.

Definition 3. Let $\mathscr{B}(U \times [0,\infty))$ be the σ-algebra of Borel subsets of $U \times [0,\infty)$. An *admissible relaxed control* or simply a *relaxed control* $m(\cdot)$ is a measure on $\mathscr{B}(U \times [0,\infty))$ such that $m(U \times [0,t]) = t$ for all t. Given a relaxed control $m(\cdot)$, there is an $m_t(\cdot)$ such that $m(d\alpha\,\mathrm{d}t) = m_t(d\alpha)\,\mathrm{d}t$. In fact, we can define $m_t(A) = \lim_{\delta\to 0} m(A \times [t-\delta, t])/\delta, \forall A \in \mathscr{B}(U)$. where $\mathscr{B}(U)$ is the σ-algebra of Borel subsets of U.

Note that $m_t(\cdot)$ is a probability measure on $\mathscr{B}(U)$. Loosely, it is the time derivative of $m(\cdot)$. It is natural to define the relaxed control representation $m^h(\cdot)$ of $u^h(\cdot)$ by $m_t^h(A) = I_{\{u^h(t)\in A\}}, \forall A \in \mathscr{B}(U)$. Let $\mathscr{F}_t^h$ denote the minimal σ-algebra that measures $\{\xi^h(s), \alpha^h(\cdot), m_s^h(\cdot), w^h(s), z^h(s), s \leqslant t\}$. Use Γ^h to denote the set of admissible relaxed controls $m^h(\cdot)$ with respect to $(\alpha^h(\cdot), w^h(\cdot))$ such that $m_t^h(\cdot)$ is a fixed probability measure in the interval $[t_n^h, t_{n+1}^h)$ given $\mathscr{F}_t^h$. Then Γ^h is a larger control space containing $\mathscr{U}^h$. With the notion of relaxed control given above, we can write (17), (14), and the value function (9) as

$$\xi^h(t) = x + \int_0^t \int_U b(\xi^h(s), \alpha^h(s), r) m_s^h(\mathrm{d}r)\,\mathrm{d}s + \int_0^t \sigma(\xi^h(s), \alpha^h(s))\,\mathrm{d}w^h(s) + \varepsilon^h(t), \tag{18}$$

$$W^h(x, \iota, m^h) = E_{x,\iota}^{m^h} \int_0^{\tau^h} \int_U \widetilde{k}(\xi^h(s), \alpha^h(s), r) m_s^h(\mathrm{d}r)\,\mathrm{d}s + E_{x,\iota}^{m^h} g(\xi^h(\tau_h), \alpha^h(\tau_h)), \tag{19}$$

$$V^h(x, \iota) = \inf_{m^h \in \Gamma^h} W^h(x, \iota, m^h). \tag{20}$$

The introduction of relaxed controls makes the control appear essentially linearly in the dynamics and cost function. In fact, the infimum of the cost over the classes of ordinary controls and relaxed controls are the same. We can rewrite (1) and (3) as

$$x(t) = x + \int_0^t \int_U b(x(s), \alpha(s), r) m_s(\mathrm{d}r)\,\mathrm{d}s + \int_0^t \sigma(x(s), \alpha(s))\,\mathrm{d}w(s), \tag{21}$$

$$W(x, \iota, m) = E_{x,\iota}^{m} \left[\int_0^{\tau} \int_U \widetilde{k}(x(s), \alpha(s), r) m_s(\mathrm{d}r)\,\mathrm{d}s + g(x(\tau), \alpha(\tau)) \right]. \tag{22}$$

Definition 4. By a weak solution of (21), we mean that there exists a probability space $(\Omega, \mathscr{F}, P)$, a filtration $\mathscr{F}_t$, and processes $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot))$ such that $w(\cdot)$ is a standard $\mathscr{F}_t$-Wiener process, $\alpha(\cdot)$ is a Markov chain with generator Q and state space $\mathscr{M}$, $m(\cdot)$ is admissible with respect to $(w(\cdot), \alpha(\cdot))$, $x(\cdot)$ is $\mathscr{F}_t$-adapted, and (21) is satisfied. For an initial condition (x, ι), by weak sense uniqueness, we mean that the probability law of admissible process $(\alpha(\cdot), m(\cdot), w(\cdot))$ determines the probability law of solution $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot))$ to (21), irrespective of the probability space.

We need two more assumptions. (A5) is a broad condition that is satisfied in most applications. The main purpose is to avoid the *tangency* problem; see Kushner and Dupuis (2001, p. 278).

(A4) Let $u(\cdot)$ be an admissible ordinary control with respect to $(w(\cdot), \alpha(\cdot))$, and suppose that $u(\cdot)$ is piecewise constant and takes only a finite number of values. Then for each initial condition, there exists a solution to (21) where $m(\cdot)$ is the relaxed control representation of $u(\cdot)$, and this solution is unique in the weak sense.

(A5) Let $\hat{\tau}(\phi) = \infty$, if $\phi(t) \in G^0$, for all $t < \infty$, otherwise, define $\hat{\tau}(\phi) = \inf\{t : \phi(t) \notin G^0\}$. The function $\hat{\tau}(\cdot)$ is continuous (as a map from $D[0, \infty)$, the space of functions that are right continuous and have left limits endowed with the Skorohod topology, to the interval $[0, \infty]$ that is compactified) with probability one relative to the measure induced by any solution to (21) with initial condition (x, ι).

6. Convergence

Consider the Markov chain $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ with transition probabilities defined in (13). Using relaxed control representation, its interpolated process $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot)\}$ can be represented by (18). We also obtain $V^h(x, \iota)$ the approximation of the value functions in (20) by dynamic programming equation. In this section, we will show that any weakly convergent subsequence of $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot)\}$ has the weak limit, denoted by $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot))$, which satisfies (21). Also the associated cost functions $W^h(x, \iota, m^h)$ converge to $W(x, \iota, m)$ of (8). Together with the tightness result, we can show that the value functions $V^h(x, \iota)$ given in (20), which is the same as (9), converge to $V(x, \iota)$ in (4). The proof of the next lemma can be obtained similar to that of Yin, Zhang, and Badowski (2003, Theorem 3.1).

Lemma 5. *Using the transition probabilities* $\{p^h(\cdot)\}$ *defined in* (13), *the interpolated process of the constructed Markov chain* $\{\alpha^h(\cdot)\}$ *converges weakly to* $\alpha(\cdot)$, *the Markov chain with generator* $Q = (q_{\iota\ell})$.

The main results are given below. The proofs are provided in the appendix to facilitate the continuity of presentation.

Theorem 6. *Assume* (A1) *and* (A2). *Let the approximating chain* $\{\xi_n^h, \alpha_n^h, n < \infty\}$ *be constructed with transition probabilities defined in* (13), $\{u_n^h, n < \infty\}$ *be a sequence of admissible controls*, $(\xi^h(\cdot), \alpha^h(\cdot))$ *be the continuous-time interpolation defined in* (6), $m^h(\cdot)$ *be the relaxed control representation of* $\{u_n^h, n < \infty\}$, *and* $\{\tilde{\tau}_h\}$ *be a sequence of* $\mathscr{F}_t^h$*-stopping times. Then* $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot), \tilde{\tau}_h\}$ *is tight. Denote the limit of a weakly convergent subsequence by* $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot), \tilde{\tau})$ *and denote by* $\mathscr{F}_t$ *the σ-algebra generated by* $\{x(s), \alpha(s), m(s), w(s), s \leqslant t, \tilde{\tau} I_{\{\tilde{\tau} \leqslant t\}}\}$. *Then* $w(\cdot)$ *is a standard* $\mathscr{F}_t$*-Wiener process*, $\tilde{\tau}$ *is an* $\mathscr{F}_t$*-stopping time, and* $m(\cdot)$ *is an admissible control. Moreover*, (21) *is satisfied.*

We next treat the convergence of the costs $W^h(x, \iota, m^h)$ given by (19), where $m^h(\cdot)$ is a sequence of admissible relaxed controls for $(\xi^h(\cdot), \alpha^h(\cdot))$. By virtue of Theorem 6, with the use of first exit time τ_h instead of $\tilde{\tau}_h$, each sequence $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot), \tau_h\}$ has a weakly convergent subsequence whose limit process satisfies (21). With a slight abuse of notation, still index the convergent subsequence by h with the limit denoted by $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot), \tilde{\tau})$. By assumption (A2), $\{\tau_h\}$ is uniformly integrable. By the weak convergence (see Theorem 6) and the Skorohod representation, as $h \to 0$,

$$E_{x,\iota}^{m^h} \int_0^{\tau^h} \int_U \widetilde{k}(\xi^h(s), \alpha^h(s), r) m_s^h(\mathrm{d}r)\,\mathrm{d}s \to E_{x,\iota}^{m} \int_0^{\tilde{\tau}} \int_U \widetilde{k}(x(s), \alpha(s), r) m_s(\mathrm{d}r)\,\mathrm{d}s,$$

$$E_{x,\iota}^{m^h} g(\xi^h(\tau_h), \alpha^h(\tau_h)) \to E_{x,\iota}^{m} g(x(\tilde{\tau}), \alpha(\tilde{\tau})). \tag{23}$$

Assumption (A5) guarantees that the first exit time of $x(\cdot)$ from G^0 is $\tilde{\tau}=\tau$. This leads to $W^h(x,\imath,m^h)\to W(x,\imath,m)$ as $h\to 0$.

Theorem 7. *Assume* (A1)–(A5). $V^h(x,\imath)$ *and* $V(x,\imath)$ *are value functions defined in* (20) *and* (4), *respectively. Then* $V^h(x,\imath)\to V(x,\imath)$ as $h\to 0$.

7. Extensions

7.1. Regime-switching jump diffusion processes

Here we consider the optimal control problem for (3) subject to a controlled regime-switching jump diffusion given by

$$\mathrm{d}x(t)=b(x(t),\alpha(t),u(t))\,\mathrm{d}t+\sigma(x(t),\alpha(t))\,\mathrm{d}w(t)+\mathrm{d}J,$$
$$J(t)=\int_0^t\int_{\Upsilon}q(x(s^-),\alpha(s),\rho)N(\mathrm{d}s,\mathrm{d}\rho), \tag{24}$$

where $N(\cdot)$ is a Poisson measure with intensity $\lambda\,\mathrm{d}t\times\Pi(\mathrm{d}\rho)$ (see the details in Kushner & Dupuis, 2001, Section 1.5), $\Pi(\cdot)$ has a compact support Υ, $q(\cdot)$ is a bounded and measurable function, and $q(\cdot,\imath,\rho)$ is continuous for each ρ and each $\imath\in\mathscr{M}$. There is an equivalent way to define the process (24) by working with the jump times and values directly. To this end, let $v_0=0$ and v_n, $n\geqslant 1$, be the time of the nth jump, and $q(\cdot,\cdot,\rho_n)$ is the corresponding jump intensity with a suitable function $q(\cdot)$. Let $\{v_{n+1}-v_n,\rho_n,n<\infty\}$ be mutually independent random variables with $v_{n+1}-v_n$ being exponentially distributed with mean $1/\lambda$, and let ρ_n have a distribution $\Pi(\cdot)$. In addition, for each n, let $\{v_{k+1}-v_k,\rho_k,k\geqslant n\}$ be independent of $\{x(s),\alpha(s),s<v_n,v_{k+1}-v_k,\rho_k,k<n\}$. Then the nth jump of the process $x(\cdot)$ is $q(x(v_n^-),\alpha(v_n),\rho_n)$, and the jump term can be written as $J(t)=\sum_{v_n\leqslant t}q(x(v_n^-),\alpha(v_n),\rho_n)$. The associated differential operator is $L^r\phi(x,\imath)=\phi_x(x,\imath)b(x,\imath,r)+\frac{1}{2}\phi_{xx}(x,\imath)\sigma^2(x,\imath)+Q\phi(x,\cdot)(\imath)+\lambda\int_{\Upsilon}[\phi(x+q(x,\imath,\rho),\imath)-\phi(x,\imath)]\Pi(\mathrm{d}\rho)$ for $\phi(\cdot,\imath)\in C^2(\mathbb{R})$. We note the following local properties of jumps for (24). Because $v_{n+1}-v_n$ is exponentially distributed, we can write $P\{x(\cdot)$ jumps on $[t,t+\Delta)|x(s),\alpha(s),w(s),N(s,\cdot),s\leqslant t\}=\lambda\Delta+o(\Delta)$. For any $H\in\mathscr{B}(\mathbb{R})$, define $\overline{\Pi}(\cdot)$ as $\overline{\Pi}(x,\imath,H)=\Pi(\rho:q(x,\imath,\rho)\in H)$. By the independence and the definition of ρ_n, $P\{x(t)-x(t^-)\in H|t=v_n;w(s),x(s),\alpha(s),N(s,\cdot),s<t;x(t^-)=x,\alpha(t)=\alpha\}=\Pi(\rho:q(x(t^-),\alpha(t),\rho)\in H)=\overline{\Pi}(x(t^-),\alpha(t),H)$. It is implied by the above discussion that the regime-switching jump diffusion $x(\cdot)$ satisfying (24) can be viewed as a process that evolves as a regime-switching diffusion (1) with jumps that occur at random time according to the jump rate defined above.

Given that the nth jump occurs at time v_n, we construct its values according to the conditional probability law, or, equivalently write it as $q(x(v_n^-),\alpha(v_n),\rho_n)$. Then the process given in (24) is a switching diffusion process in (1) until the time of the next jump. Hence, the approximating Markov chain $\{(\xi_n^h,\alpha_n^h)\}$ with jump process can be constructed in an analogous way to the previous section. Suppose that the current state is $\xi_n^h=x,\alpha_n^h=\imath$ and control is $u_n^h=r$. The next interpolation interval $\Delta t^h(x,\imath,r)$ is determined by (13). Then we determine the next state $(\xi_{n+1}^h,\alpha_{n+1}^h)$ by noting: (a) No jumps occur in $[t_n^h,t_{n+1}^h)$ with probability $(1-\lambda\Delta t^h(x,\imath,r)+o(\Delta t^h(x,\imath,r))$; we determine $(\xi_{n+1}^h,\alpha_{n+1}^h)$ by transition probability $p_D^h(\cdot)$ as in (13). (b) There is a jump in $[t_n^h,t_{n+1}^h)$ with probability $\lambda\Delta t^h(x,\imath,r)+o(\Delta t^h(x,\imath,r)$; we determine $(\xi_{n+1}^h,\alpha_{n+1}^h)$ by $\xi_{n+1}^h=\xi_n^h+q_h(x,\imath,\rho)$, $\alpha_{n+1}^h=\alpha_n^h$, where $\rho\sim\Pi(\cdot)$, and $q_h(x,\imath,\rho)\in S_h\subseteq\mathbb{R}$ such that $q_h(x,\imath,\rho)$ is the nearest value of $q(x,\imath,\rho)$ so that $\xi_{n+1}^h\in S_h$. Then $|q_h(x,\imath,\rho)-q(x,\imath,\rho)|\to 0$ as $h\to 0$, uniformly in x. Let H_n^h denote the event that $(\xi_{n+1}^h,\alpha_{n+1}^h)$ is determined by the first case above and use T_n^h to denote the event of the second case. Let $I_{H_n^h}$ and $I_{T_n^h}$ be corresponding indicator functions, respectively. Then $I_{T_n^h}=1-I_{H_n^h}$. We also need a new definition of local consistency for Markov chain approximation of the regime-switching jump diffusions.

Definition 8. A controlled Markov chain $\{(\xi_n^h,\alpha_n^h),n<\infty\}$ is said to be *locally consistent* with (24), if there is an interpolation interval $\Delta t^h(x,\imath,r)\to 0$ as $h\to 0$ uniformly in x, $\imath$, and r such that (a) there is a transition probability $p_D^h(\cdot)$ that (together with an interpolation interval Δt^h) is locally consistent with (1) in the sense that (7) holds; (b) there is a $\delta^h(x,\imath,r)=o(\Delta t^h(x,\imath,r))$ such that the one-step transition probability $p^h((x,\imath),(y,\ell)|r)$ is given by $p^h((x,\imath),(y,\ell)|r)=(1-\lambda\Delta t^h(x,\imath,r)-\delta^h(x,\imath,r))p_D^h((x,\imath)(y,\ell)|r)+(\lambda\Delta t^h(x,\imath,r)+\delta^h(x,\imath,r))\Pi\{\rho:q_h(x,\imath,\rho)=y-x\}$.

Local consistency can be easily verified with the use of the local properties of jumps specified. To proceed, let the discrete times at which jumps occur be denoted by v_j^h, $j=1,2,\ldots$. Then $\xi_{v_j^h}^h-\xi_{v_j^h-1}^h=q_h(\xi_{v_j^h-1}^h,\alpha_{v_j^h-1}^h,\rho)$. Let $\xi_0^h=x,\alpha_0^h=\imath$, and E_n^h be the expectation conditioned on the data up to time n (conditioned on $\mathscr{D}_n^h$, the σ-algebra generated by $\{\xi_k^h,\alpha_k^h,u_k^h,H_k^h,k\leqslant n;v_k^h,\rho_k\ :\ v_k^h<t_n\}$). Then we can write $\xi_n^h=x+\sum_{k=0}^{n-1}\Delta\xi_k^h(1-I_{T_k^h})+\sum_{k=0}^{n-1}\Delta\xi_k^h I_{T_k^h}=x+\sum_{k=0}^{n-1}E_k^h\Delta\xi_k^h(1-I_{T_k^h})+\sum_{k=0}^{n-1}(\Delta\xi_k^h-E_k^h\Delta\xi_k^h)(1-I_{T_k^h})+\sum_{k:v_k<n}q_h(\xi_{v_k}^h,\alpha_{v_k}^h,\rho_k)$. We use M_n^h and J_n^h to denote the third and fourth terms in the last equation. By the local consistency, $\xi_n^h=x+\sum_{k=0}^{n-1}[b(\xi_k^h,\alpha_k^h,u_k^h)\Delta t_k^h+o(\Delta t_k^h)]+(\max_{k_1\leqslant n}\Delta t_{k_1}^h)O(\sum_{k=0}^{n-1}I_{T_k^h})+M_n^h+J_n^h$, where M_n^h is a martingale with respect to $\mathscr{D}_n^h$ whose quadratic variation is given by $\sum_{k=0}^{n-1}(\sigma^2(\xi_k^h,\alpha_k^h)\Delta t_k^h+o(\Delta t_k^h))(1-I_{T_k^h})=\sum_{k=0}^{n-1}(\sigma^2(\xi_k^h,\alpha_k^h)\Delta t_k^h+o(\Delta t_k^h))+(\max_{k_1\leqslant n}\Delta t_{k_1}^h)O(\sum_{k=0}^{n-1}I_{T_k^h})$. Note that $E\sum_{k=0}^{n-1}I_{T_k^h}=E[\text{number of } n\ :\ v_n^h\leqslant t]\to\lambda t=O(1)$ as $h\to 0$. This implies $(\max_{k_1\leqslant n}\Delta t_{k_1}^h)O(\sum_{k=0}^{n-1}I_{T_k^h})\to 0$ in probability as $h\to 0$. So we can drop the term involving $I_{T_k^h}$ without affecting the limit, obtain $\xi^h(t)=x+\int_0^t b(\xi^h(s),\alpha^h(s),u(s))\,\mathrm{d}s+M^h(t)+J^h(t)+\varepsilon^h(t)$,where $J^h(t)=\sum_{v_n^h\leqslant t}q_h(\xi^h(v_n^{h}-),\alpha^h(v_n^h),\rho_n)$, and $\varepsilon^h(t)$ is used as in (15) and $M^h(\cdot)$ is martingale with quadratic variation $\int_0^t\sigma^2(x^h(s),\alpha^h(s))\,\mathrm{d}s+\varepsilon^h(t)$. For practical purpose of computation, modify the system of dynamic programming equations as $V^h(x,\imath)=\min_{r\in U^h}[[1-\lambda\Delta t^h(x,\imath,r)-$

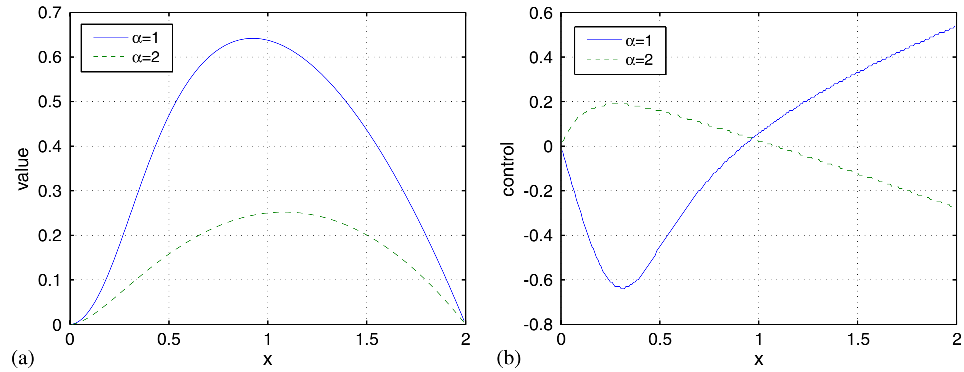

Fig. 1. Iteration in policy space with $h=2^{-9}$, Example 9. (a) appr. value function; (b) optimal feedback control.

$\delta^h(x,\imath,r)]\sum_{(y,\ell)} p_D^h((x,\imath),(y,\ell)|r)V^h(y,\ell)$ $+[\lambda\Delta t^h(x,\imath,r)+\delta^h(x,\imath,r)]\int_\Upsilon V^h(x+q_h(x,\imath,\rho),\imath)\Pi(\mathrm{d}\rho)+\widetilde{k}(x,\imath,r)\Delta t^h(x,\imath,r)]$ with boundary conditions $V^h(x,\imath)=g(x,\imath)$, $x\in\partial G$. Using analogous approach as in previous section, we can obtain convergence result.

7.2. Discounted cost problems

In many applications, it is desirable to deal with discounted cost problems in an infinite-time horizon. To construct numerical algorithms for controlled regime-switching jump diffusions, the state space for the x-component needs to be bounded. Thus, we adopt the ideas in Kushner (1990) and still set this up as a stopping time problem. Consider a discounted cost problem subject to (24). The cost function is given by

$$W(x,\imath,u)=E^u_{x,\imath}\left[\int_0^\tau e^{-\beta s}\widetilde{k}(x(s),\alpha(s),u(s))\,\mathrm{d}s + g(x(\tau),\alpha(\tau))\right], \tag{25}$$

where $\beta>0$ is the discount rate. Stopping time τ and value function $V(x,\imath)$ are defined as in (1) and (4), respectively. Then, $V(x,\imath)$ satisfies the system of HJB equations $\inf_{r\in U}[L^r V(x,\imath)-\beta V(x,\imath)+\widetilde{k}(x,\imath,r)]=0$, for all $(x,\imath)\in G^0\times\mathscr{M}$ with boundary condition $V(x,\imath)=g(x,\imath)$ for $x\in\partial G$. Hence, we can compute the value similar to the method for solving (3) subject to (24). The system of dynamic programming equations is given by $V^h(x,\imath)=\min_{r\in U}[(1-\lambda\Delta t^h(x,\imath,r)-\delta^h(x,\imath,r))\sum_{(y,\ell)}e^{-\beta\Delta t^h(x,\imath,r)}p_D^h((x,\imath),(y,\ell)|r)V^h(y,\ell)+(\lambda\Delta t^h(x,\imath,r)+\delta^h(x,\imath,r))e^{-\beta\Delta t^h(x,\imath,r)}\int_\Upsilon V^h(x+q_h(x,\imath,\rho),\imath)\Pi(\mathrm{d}\rho)+\widetilde{k}(x,\imath,r)\Delta t^h(x,\imath,r)]$. Using the method leading to (13) from (12), and approximating $e^{-\beta\Delta t^h(x,\imath,r)}$ by $1-\beta\Delta t^h(x,\imath,r)$, we have the same transition probability as in (13), but slightly different time interval function $\Delta t^h(x,\imath,r)=h^2/(\widetilde{D}+\beta h^2)$. It can be demonstrated as in the previous sections, the approximation so constructed is consistent. In addition, the convergence results carry over to the current setup.

8. Examples

In this section, we provide several examples for demonstration. All the numerical experiments were computed using MATLAB on a WinXP platform.

Example 9. Consider a LQ regulator system with regime-switching. The dynamic system is given by $\mathrm{d}x(t)=A(\alpha(t))x(t)\,\mathrm{d}t+B(\alpha(t))u(t)\,\mathrm{d}t+C(\alpha(t))x(t)\,\mathrm{d}w(t)$, where the control $u(\cdot)$ takes value in a subset of $\mathbb{R}$; the Markov chain $\alpha(\cdot)\in\mathscr{M}$ with $\mathscr{M}=\{1,2\}$ and generator Q, a 2×2 matrix with two columns $(-0.5,0.5)'$ and $(0.5,-0.5)'$. The set G is $[0,2]$. Coefficients are $A(\imath)=B(\imath)=3-2\imath$ and $C(\imath)=\imath$. The cost function is defined as $W(x,\imath,u)=E^u_{x,\imath}\int_0^\tau(x^2(t)+u^2(t))\,\mathrm{d}t$ and the value function is $V(x,\imath)=\inf_{u(\cdot)}W(x,\imath,u)$. Using the algorithms developed in this paper in conjunction with iteration in policy space, we can obtain $V_n^h(\cdot)\to V^h(\cdot)$ as $n\to\infty$, where $V^h(\cdot)$ is the value function in (9). The procedure is outlined as: (1) for a pre-specified tol>0: set $n=0$, for $(x,\imath)\in G_h^0\times\mathscr{M}$, take initial control $u_0^h(x,\imath)=1$. With r replacing by corresponding $u_0^h(x,\imath)$, solve (11) to find $V_0^h(\cdot)$;

(2) find an improved control by $u_{n+1}^h(x,\imath):=\arg\min_{r\in U^h}[\sum_{y,\ell}p^h((x,\imath),(y,\ell)|r)V_n^h(y,\ell)+\widetilde{k}(x,\imath,r)\Delta t^h(x,\imath,r)]$;

(3) find $V_{n+1}^h(\cdot)$ with $u_{n+1}^h(\cdot)$ by solving (11). If $|V_{n+1}^h-V_n^h|>$ tol, then go to step (2) with $n\to n+1$.

Fig. 1 is the result of approximation of value and optimal control for each initial status in G_h^0 when step-size is $h=2^{-9}$. Solid line and dotted line are for $\alpha=1$ and $\alpha=2$, respectively. Table 1 presents the approximated values for selected initial states in comparison with various step-sizes. The iterative scheme was applied until the maximum difference between successive iterates was less than tol$=10^{-6}$.

Example 10. Consider a modified version of the model studied in Ghosh, Arapostathis, and Marcus (1993) (*a flexible manufacturing system*). Suppose there is one machine producing a single commodity. The inventory $x(t)\in[0,2]$ is governed by $\mathrm{d}x(t)=(u(t)-d(\alpha(t)))\,\mathrm{d}t+\sigma(\alpha(t))\,\mathrm{d}w(t)+\mathrm{d}J(t)$, where $\alpha(t)$ represents marketing state (a continuous-time Markov chain having generator Q with $q_{11}=q_{22}=-0.5$ and $q_{21}=q_{12}=0.5$),

Table 1
Values with various initial vs h for Example 9

Initial state	Step size h					
	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
(0.5, 1)	0.4724	0.4707	0.4697	0.4691	0.4688	0.4686
(0.5, 2)	0.1625	0.1601	0.1587	0.1580	0.1577	0.1575
(1, 1)	0.6252	0.6322	0.6358	0.6375	0.6384	0.6389
(1, 2)	0.2513	0.2508	0.2505	0.2503	0.2502	0.2501
(1.5, 1)	0.4280	0.4326	0.4350	0.4362	0.4368	0.4371
(1.5, 2)	0.2004	0.2006	0.2006	0.2006	0.2006	0.2006

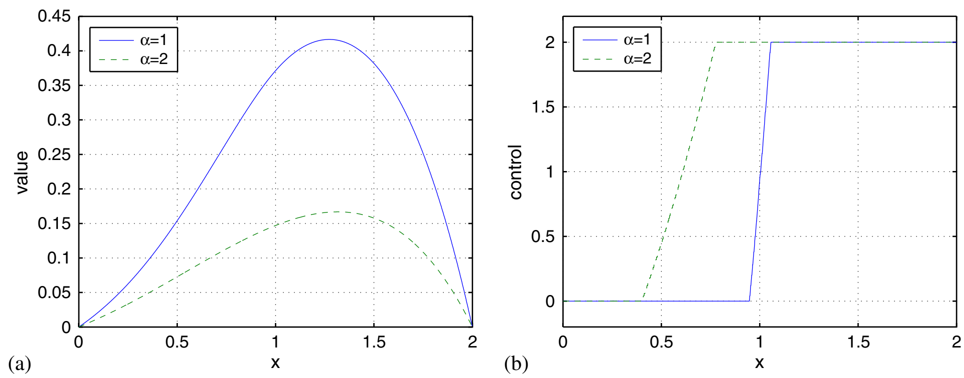

Fig. 2. Iteration in policy space with $h = 2^{-10}$, Example 10. (a) appr. value function; (b) optimal feedback control.

Table 2
Values with various initial vs h for Example 10

Initial state	Step size h			
	2^{-7}	2^{-8}	2^{-9}	2^{-10}
(0.5, 1)	0.1528	0.1534	0.1539	0.1539
(0.5, 2)	0.0725	0.0726	0.0727	0.0727
(1, 1)	0.3697	0.3706	0.3712	0.3713
(1, 2)	0.1468	0.1470	0.1470	0.1471
(1.5, 1)	0.3814	0.3816	0.3817	0.3819
(1.5, 2)	0.1577	0.1577	0.1577	0.1577

taking values in $\mathscr{M}=\{1,2\}$, and $d(\cdot)$ is the demand rate depending on market with $d(1)=1$ and $d(2)=2$. The production rate $u(t)$ taking its value in $[0, 2]$ is a control parameter, $\sigma\,\mathrm{d}w(t)$ is interpreted as minor demand fluctuation with $\sigma(1)=1$ and $\sigma(2)=2$, and $J(t)$ is a Poisson jump process that is interpreted as sales returns with $J(t)=\sum_{v_n \leqslant t}\rho_n$, where $\rho_n \in \Upsilon = \{0.01, 0.02\}$, with its distribution $\Pi(0.01)=0.6$, $\Pi(0.02)=0.4$. Let $\lambda=4$, and the $\{v_{n+1}-v_n\}$ is a sequence of exponentially distributed random variables with mean $1/\lambda$. The cost function is $W(x,\imath,u)=E^u_{x,\imath}\int_0^\tau (x(t)-0.2u(t))^2\,\mathrm{d}t$, and the value function is $V(x,\imath)=\inf_{u(\cdot)} W(x,\imath,u)$. Policy iteration used for this example is similar to that of Example 9. The main difference is that the system of dynamic programming equations used is the one developed in Section 7.1. Fig. 2 and Table 2 present the computation results.

9. Further remarks

This paper is devoted to numerical methods for approximating regime-switching diffusions and regime-switching jump-diffusions. For notational simplicity, the problem is setup such that the x-component of the state is a scalar-valued function. The results obtained readily extend to systems with multi-dimensional diffusion processes.

For a regime-switching system in which the Markov chain has a large state space, we may use the ideas of two-time-scale approach presented in Yin and Zhang (1998) (see also Yin & Zhang, 2005 and references therein) to first reduce the complexity of the underlying system and then construct numerical solutions for the limit systems. As demonstrated in the aforementioned references, the limit control problems can be used for construction of controls of the original systems leading to near optimality. It would be interesting to obtain rate of convergence for the numerical method developed in this paper. For stochastic control problems without switching, the rate of convergence has only become available very recently; see Krylov (2000). The essence is the use of nonlinear PDE (partial differential equation) techniques. The addition of switching adds another fold of complication, namely, one needs to deal with nonlinear system of PDEs. Another important problem concerns the problem when the diffusion term is also controlled. The complication here is that the set of models is not “closed” under convergence of controls (even relaxed controls). A remedy is to enlarge the set of models by introducing “martingale measure driving processes;" see Kushner (1990). This does not change the optimal cost or controls, but facilitates the convergence proof and approximations. The main problems are the consistency issue and the construction of easily codable algorithms; see Kushner (2000). This problem is our current research project.

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Appendix A. Proofs of results

Proof of Theorem 6. Define a topology for the set $[0, \infty]$ such that this set is compact (via the compactification). Then the sequences $\{\alpha^h(\cdot), m^h(\cdot), \tilde{\tau}_h\}$ are always tight, since their range spaces are compact. Thus, owing to Billingsley (1968, Theorem 7.7, p.48), it suffices to prove that the tightness of $\{w^h(\cdot)\}$ and $\{\xi^h(\cdot)\}$. By the local consistency and the definition of $w^h(\cdot)$ in (16), we obtain

$$E(w^h(t+s) - w^h(t))^2 = s + \varepsilon^h(s), \tag{A.1}$$

where $\varepsilon^h(\cdot)$ is a continuous function defined in (15). Let $\mathscr{T}^h_T$ be the set of $\mathscr{F}^h_t$-stopping times being less than or equal to T w.p.1. Then for $\delta > 0$, $\tilde{v}_h \in \mathscr{T}^h_T$, by (A.1) and the strong Markov property,

$$E^{m^h}_{\tilde{v}_h} |w^h(\tilde{v}_h + \delta) - w^h(\tilde{v}_h)|^2 = \delta + \varepsilon^h(\delta), \tag{A.2}$$

where $E^{m^h}_{\tilde{v}_h}$ is the conditional expectation with respect to $\mathscr{F}^h_{\tilde{v}_h}$. Taking $\limsup_{h\to 0}$ followed by $\lim_{\delta\to 0}$ yield the tightness of $\{w^h(\cdot)\}$.

Next we prove the tightness of $\{\xi^h(\cdot)\}$. Let $E^h_{x,\imath}$ be the expectation for the interpolated process with interpolation step size h and initial data $(x, \imath)$. By (A1), (18), and (A.1), we have $E^h_{x,\imath}|\xi^h(t) - x|^2 = E^h_{x,\imath}| \int_0^t \int_U b(\xi^h(s), \alpha^h(s), r) m^h_s(\mathrm{d}r)\,\mathrm{d}s + \int_0^t \sigma(\xi^h(s), \alpha^h(s))\ \mathrm{d}w^h(s) + \varepsilon^h(t)|^2 = 3E^h_{x,\imath}| \int_0^t \int_U b(\xi^h(s), \alpha^h(s), r) m^h_s(\mathrm{d}r)\ \mathrm{d}s|^2 + 3E^h_{x,\imath}| \int_0^t \sigma(\xi^h(s), \alpha^h(s))\,\mathrm{d}w^h(s)|^2 + \varepsilon^h(t) \leqslant Kt^2 + Kt + \varepsilon^h(t)$, where K is a generic positive constant. Applying similar argument as that of (A.2), we also have $E^{m^h}_{\tilde{v}_h}|\xi^h(\tilde{v}_h + \delta) - \xi^h(\tilde{v}_h)|^2 = O(\delta) + \varepsilon^h(\delta)$, as $\delta \to 0$. This shows the tightness of $\xi^h(\cdot)$. So far, we have proved that $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot), \tilde{\tau}_h\}$ is tight.

For the rest of the proof, we assume the probability space is chosen as required by Skorohod representation (see Kushner & Dupuis (2001, Theorem 1.7, Chapter 9)). With a slight abuse of notation, we assume the convergence of the sequence $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), w^h(\cdot), \tilde{\tau}_h\}$ itself with limit denoted by $(x(\cdot), \alpha(\cdot), m(\cdot), w(\cdot), \tilde{\tau})$, and the convergence is in the sense of w.p.1 via Skorohod representation. To characterize $w(\cdot)$, let $t > 0$, $\delta > 0$, p, q, $\{t_k : k \leqslant p\}$ be given such that $t_k \leqslant t \leqslant t + r$ for all $k \leqslant p$, $P(\tilde{\tau}_h = t_k)$ is zero, $\phi_j(\cdot)$ for $j \leqslant q$ is real-valued and continuous functions on $U \times [0, \infty)$ and having compact support for all $j \leqslant q$. Define $(\phi_j, m)_t \overset{\text{def}}{=} \int_0^t \int_U \phi_j(r, s) m(\mathrm{d}r\,\mathrm{d}s)$. Let $H(\cdot)$ be a real-valued and continuous function of its arguments with compact support. By (16), $w^h(\cdot)$is an $\mathscr{F}^h_t$-martingale. Thus we obtain $EH(\xi^h(t_k), \alpha^h(t_k), w^h(t_k), (\phi_j, m^h)_{t_k}, j \leqslant q, k \leqslant p, \tilde{\tau}_h I_{\{\tilde{\tau}_h \leqslant t\}})$ $[w^h(t + r) - w^h(t)] = 0$. Using the Skorohod representation and the dominant convergence theorem, passing to the limits as $h \to 0$, we have $EH(x(t_k), \alpha(t_k), w(t_k), (\phi_j, m)_{t_k}, j \leqslant q, k \leqslant p, \tilde{\tau} I_{\{\tilde{\tau} \leqslant t\}})(w(t + r) - w(t)) = 0$. Since $w(\cdot)$ has continuous sample paths, this implies that $w(\cdot)$ is a continuous $\mathscr{F}_t$-martingale. Note that $E[(w^h(t + \delta))^2 - (w^h(t))^2] = E[(w^h(t+\delta) - w^h(t))^2]$. Again, by the Skorohod representation and the dominant convergence theorem together with (A.2), we have $EH(x(t_k), \alpha(t_k), w(t_k), (\phi_j, m)_{t_k}, j \leqslant q, k \leqslant p, \tilde{\tau} I_{\{\tilde{\tau} \leqslant t\}})$ $(w^2(t + \delta) - w^2(t) - \delta) = 0$. The quadratic variation of the martingale $w(t)$ is t, which implies $w(\cdot)$ is an $\mathscr{F}_t$-Wiener process. For $\delta > 0$ and any process $y(\cdot)$, define the process $y^\delta(\cdot)$ by $y^\delta(t) = y(n\delta)$, $t \in [n\delta, n\delta + \delta)$. Then, by the tightness of $\{\xi^h(\cdot), \alpha^h(\cdot)\}$, (18) can be written as $\xi^h(t) = x + \int_0^t \int_U b(\xi^h(s), \alpha^h(s), r) m^h_s(\mathrm{d}r)\,\mathrm{d}s + \int_0^t \sigma(\xi^{h,\delta}(s), \alpha^{h,\delta}(s))\,\mathrm{d}w^h(s) + \varepsilon^{h,\delta}(t)$, where $\lim_\delta \limsup_h E|\varepsilon^{h,\delta}(t)| = 0$.

Taking limit as $h \to 0$, the convergence with probability one (through Skorohod representation) yields $E| \int_0^t \int_U b(\xi^h(s), \alpha^h(s), r) m^h_s(\mathrm{d}r)\,\mathrm{d}s - \int_0^t \int_U b(x(s), \alpha(s), r)\ m^h_s(\mathrm{d}r)\,\mathrm{d}s| \to 0$ uniformly in t. On the other hand, the sequence $\{m^h(\cdot)\}$ converges in the "compact weak" topology. In particular, for any bounded and continuous function $\phi(\cdot)$ with compact support, $(\phi, m^h)_\infty \to (\phi, m)_\infty$. The weak convergence and the Skorohod representation imply that $\int_0^t \int_U b(x(s), \alpha(s), r) m^h_s(\mathrm{d}r)\,\mathrm{d}s - \int_0^t \int_U b(x(s), \alpha(s), r) m_s(\mathrm{d}r)\,\mathrm{d}s \to 0$ uniformly in t on any bounded interval with probability one. Owing to the fact that the $\xi^{h,\delta}(\cdot)$ and $\alpha^{h,\delta}(\cdot)$ are piecewise constant functions, it follows from the probability one convergence, $\int_0^t \sigma(\xi^{h,\delta}(s), \alpha^{h,\delta}(s))\,\mathrm{d}w^h(s) \to \int_0^t \sigma(x^\delta(s), \alpha^\delta(s))\,\mathrm{d}w(s)$. Combining the above results, $x(t) = x + \int_0^t \int_U b(x(s), \alpha(s), r) m_s(\mathrm{d}r)\,\mathrm{d}s + \int_0^t \sigma(x^\delta(s), \alpha^\delta(s))\,\mathrm{d}w(s) + \varepsilon^\delta(t)$, where $\lim_{\delta\to 0} E|\varepsilon^\delta(t)| = 0$. Finally, taking limits in the above equation as $\delta \to 0$ yields the result. □

Proof of Theorem 7. By (A2), $\{\tau_h\}$ is uniformly integrable. For each h, let $\widehat{m}^h$ be an optimal relaxed control for $\{\xi^h(\cdot), \alpha^h(\cdot)\}$. That is, $V^h(x, \imath) = W^h(x, \imath, \widehat{m}^h) = \inf_{m^h} W^h(x, \imath, m^h)$. Choose a subsequence $\{h'\}$ of $\{h\}$ such that $\liminf_{h\to 0} V^h(x, \imath) = \lim_{h'\to 0} V^{h'}(x, \imath) = \lim_{h'\to 0} W^{h'}(x, \imath, \widehat{m}^{h'})$ We assume $\{\xi^{h'}(\cdot), \alpha^{h'}(\cdot), w^{h'}(\cdot), \widehat{m}^{h'}(\cdot), \tau^{h'}\}$converges weakly to $(x(\cdot), \alpha(\cdot), w(\cdot), m(\cdot), \tau)$. Otherwise, take a subsequence of $\{h'\}$ to assure its weak limit. Then by (23), $W^{h'}(x, \imath, m^{h'}) \to$

$W(x,\imath,m) \geqslant V(x,\imath)$. It follows that $\liminf_h V^h(x,\imath) \geqslant V(x,\imath)$. Thus, we need only prove that $\limsup_h V^h(x,\imath) \leqslant V(x,\imath)$.

Let $\overline{m}$ be an optimal admissible control with respect to $(w(\cdot),\alpha(\cdot))$ such that $\overline{x}(\cdot)$ and $\overline{\tau}$ are the associated solution and stopping time, and $W(x,\imath,\overline{m}) = V(x,\imath)$. We need to approximate $\overline{m}(\cdot)$ in such a way that it can be applied to (ξ_n^h, α_n^h). First, note the following fact: Let $\overline{m}^\delta(\cdot)$ be an admissible relaxed control representation of piecewise constant control $\overline{u}^\delta(\cdot)$ with respect to $(w(\cdot),\alpha(\cdot))$, and let $\overline{x}^\delta(\cdot)$ and $\overline{\tau}^\delta$ be the associated solution and stopping time. Using $\Rightarrow$ to denote weak convergence, if $(\overline{m}^\delta(\cdot),\alpha(\cdot),w(\cdot)) \Rightarrow (\overline{m}(\cdot),\alpha(\cdot),w(\cdot))$, we also have $\overline{m}^\delta(\cdot),\alpha(\cdot),w(\cdot),\overline{x}^\delta(\cdot),\overline{\tau}^\delta \Rightarrow (\overline{m}(\cdot),\alpha(\cdot),w(\cdot),\overline{x}(\cdot),\overline{\tau})$, where (21) holds for the limit and $\overline{\tau}$ is the associated stopping time by Theorem 6 and (A4). Also, with assumption (A5), $W(x,\imath,\overline{m}^\delta) \to W(x,\imath,\overline{m})$, we will use this fact to approximate the $\overline{m}(\cdot)$ policy so that it can be applied to $(\xi^h(\cdot),\alpha^h(\cdot))$.

By the well-known chattering lemma, given any $\varepsilon > 0$, there is a $\delta > 0$ such that we can approximate $\overline{m}(\cdot)$ by an ordinary control $\overline{u}^\varepsilon(\cdot)$ with the following properties. (a) $\overline{u}^\varepsilon(\cdot)$ takes only finitely many values (denoted by U_ε); (b) $\overline{u}^\varepsilon(\cdot)$ is constant on the intervals $[k\delta, k\delta+\delta]$, $k = 0,1,\ldots$; (c) with $\overline{m}^\varepsilon(\cdot)$ denoting the relaxed control representation of $\overline{u}^\varepsilon(\cdot)$, we have, as $\varepsilon \to 0$, $(\overline{m}^\varepsilon(\cdot),\alpha(\cdot),w(\cdot),\overline{x}^\varepsilon(\cdot),\overline{\tau}^\varepsilon)$ converges weakly to $(\overline{m}(\cdot),\alpha(\cdot),w(\cdot),\overline{x}(\cdot),\overline{\tau})$, (d) by Kushner and Dupuis (2001), Theorem 10.3.1 $W(x,\imath,\overline{m}^\varepsilon) \leqslant V(x,\imath) + \varepsilon$.

Note that under assumption (A5) and the weak convergence, the exit times of $\overline{x}^\varepsilon(\cdot)$ on G converge to that of $\overline{x}(\cdot)$. For each $\varepsilon > 0$, and the corresponding δ in the chattering lemma, consider an optimal control problem for (3) subject to the constraint (1), but where the controls are constants over the interval $[k\delta, k\delta+\delta)$, $k = 0,1,\ldots$, and take values in U_ε. This corresponds to controlling the discrete-time Markov process that is obtained by sampling $x(\cdot)$ and $\alpha(\cdot)$ at times $k\delta$, for $k = 0,1,\ldots$ (The optimal control for this "sampled" problem is an ordinary feedback control.) Let $\hat{u}^\varepsilon(\cdot)$ denote the optimal control and $\hat{m}^\varepsilon(\cdot)$ be its relaxed control representation, and let $\hat{x}^\varepsilon(\cdot)$ denote the associated solution process. Since, $\hat{m}^\varepsilon(\cdot)$ is optimal in the chosen class of controls, we have $W(x,\imath,\hat{m}^\varepsilon) \leqslant W(x,\imath,\overline{m}^\varepsilon) \leqslant V(x,\imath) + \varepsilon$. We next approximate $\hat{u}^\varepsilon(\cdot)$ by a suitable function of $w(\cdot)$ and $\alpha(\cdot)$. By assumption (A4), $(\hat{m}^\varepsilon(\cdot),\alpha(\cdot),w(\cdot),\hat{x}^\varepsilon(\cdot))$ is determined by the initial condition $x(0) = x$, $\alpha(0) = \alpha$, and probability law of $(\hat{m}^\varepsilon(\cdot),\alpha(\cdot),w(\cdot))$. For $r \in U_\varepsilon$ define the function $F_n^{\varepsilon,\theta}$ as $F_n^{\varepsilon,\theta}(r; x, \hat{u}^\varepsilon(j\delta), j < n, w(j\theta), \alpha(j\theta), j\theta \leqslant n\delta) = P\{\hat{u}^\varepsilon(n\delta) = r | x, \hat{u}^\varepsilon(j\delta), j < n; w(j\theta), \alpha(j\theta), j\theta \leqslant n\delta\}$. We can assume the function $F_n^{\varepsilon,\theta}$ is continuous in the w-arguments and α-arguments. Otherwise, we can use similar techniques as in Kushner and Dupuis (2001), p. 285. Because the σ-algebra determined by the set $\{\hat{u}^\varepsilon(j\delta), j < n; w(j\theta), \alpha(j\theta), j\theta \leqslant n\delta\}$ increases to the σ-algebra determined by $\{\hat{u}^\varepsilon(j\delta), j < n; w(s), \alpha(s), s \leqslant n\delta\}$ as $\theta \to 0$, the martingale convergence theorem implies that for each n, r, and δ, $F_n^{\varepsilon,\theta}(r; x, \hat{u}^\varepsilon(j\delta), j < n, w(j\theta), \alpha(j\theta), j\theta \leqslant n\delta) \to P\{\hat{u}^\varepsilon(n\delta) = r | x, \hat{u}^\varepsilon(j\delta), j < n; w(s), \alpha(s), s \leqslant n\delta\}$ with probability one as $\theta \to 0$.The control $u_n^{\varepsilon,\theta}$, with its interpolated process $u^{\varepsilon,\theta}(\cdot)$, is defined by the probability law $P\{u_n^{\varepsilon,\theta} = r | x, u^{\varepsilon,\theta}(j\delta), j < n; w(s), \alpha(s), s \leqslant n\delta\} = F_n^{\varepsilon,\theta}(r; x, u^{\varepsilon,\theta}(j\delta), j < n, w(j\theta), \alpha(j\theta), j\theta \leqslant n\delta)$. Let $m^{\varepsilon,\theta}(\cdot)$ be the relaxed control representation of the ordinary control $u^{\varepsilon,\theta}(\cdot)$, $x^{\varepsilon,\theta}(\cdot)$ and $\tau^{\varepsilon,\theta}(\cdot)$ be the associated solution and stopping time. Using the convergence of $F_n^{\varepsilon,\theta}$ above, we conclude $m^{\varepsilon,\theta}(\cdot) \Longrightarrow \hat{m}^\varepsilon(\cdot)$ as $\theta \to 0$. Hence, there exist small enough θ such that $W(x,\imath,m^{\varepsilon,\theta}) \leqslant W(x,\imath,\hat{m}^\varepsilon) + \varepsilon$. We now adapt $F_k^{\varepsilon,\theta}(\cdot)$such that it can be applied to (ξ_n^h, α_n^h). For n satisfying $k\delta \leqslant t_n^h < k\delta + \delta$, define the control $\overline{u}_n^h$, with its interpolated process $\overline{u}^h(\cdot)$ (with interpolation intervals $\Delta t_n^h = \Delta t^h(\xi_n^h, \alpha_n^h, \overline{u}_n^h)$) by conditional probability law $P\{\overline{u}_n^h = r | x; \overline{u}_j^h, j \leqslant n; w^h(j\theta), \alpha^h(j\theta), j\theta \leqslant t_n^h\} = F_k^{\varepsilon,\theta}(r; x, \overline{u}^h(j\delta), j \leqslant k, w^h(j\theta), \alpha^h(j\theta), j\theta \leqslant k\delta\}$. Let $\overline{m}^h(\cdot)$ denote the relaxed control equivalent to $\overline{u}^h(\cdot)$. Then, as $h \to 0$, using Skorohod representation theorem, $(\xi^h(\cdot), \alpha^h(\cdot), \overline{m}^h(\cdot), w^h(\cdot), \tau_h) \Rightarrow (x^{\varepsilon,\theta}(\cdot), \alpha(\cdot), m^{\varepsilon,\theta}(\cdot), w(\cdot), \tau^{\varepsilon,\theta})$. By the optimality of $V^h(x,\imath)$ and the above weak convergence, $V^h(x,\imath) \leqslant W^h(x,\imath,\overline{m}^h) \to W(x,\imath,m^{\varepsilon,\theta})$. The above inequalities and the convergence yield that $\limsup_h V^h(x,\imath) \leqslant V(x,\imath) + 2\varepsilon$, for the chosen subsequence. Since any subsequence of $\{\xi^h(\cdot), \alpha^h(\cdot), w^h(\cdot), m^h(\cdot), \tau_h\}$ has a subsequence that converges weakly, and since ε is arbitrary, $\limsup_h V^h(x,\imath) \leqslant V(x,\imath)$ as desired. □

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