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NATURAL SUPERCONVERGENCE POINTS IN THREE-DIMENSIONAL FINITE ELEMENTS*

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Abstract. A systematic and analytic process is conducted to identify natural superconvergence points of high degree polynomial C^0 finite elements in a three-dimensional setting. This identification is based upon explicitly constructing an orthogonal decomposition of local finite element spaces. Derivative and function value superconvergence points are investigated for both the Poisson and the Laplace equations. Superconvergence results are reported for hexahedral, pentahedral, and tetrahedral elements up to certain degrees.

Key words. finite element methods, natural superconvergence, hexahedral, pentahedral (triangular prism), tetrahedral elements polynomial

AMS subject classifications. 65N30, 65M60, 74S05, 41A10

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1. Introduction.

In this paper, we study *natural superconvergence* points of the finite element (FE) approximation u_h to the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ \quad u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is an open bounded domain with Lipschitz boundary $\partial\Omega$ and f and g are sufficiently smooth given functions. When $f \equiv 0$, the Poisson equation (1.1) reduces to the Laplace equation.

Superconvergence for FE methods has been a research topic for about 35 years. For the literature, the reader is referred to [1, 2, 4, 9, 10, 14, 15, 16, 22, 26, 27] and references therein. Here, by "natural superconvergence," we mean that the superconvergence points are obtained without employing any postprocessing in FE solutions. The authors are aware that many superconvergence results can be obtained from postprocessing or recovery techniques. Nevertheless, our investigation is limited to the natural superconvergence in this paper. Moreover, *natural superconvergence* is abbreviated as *superconvergence* when no confusion arises.

Superconvergence points for the three-dimensional (3D) tensor-product FE can be determined from one-dimensional (1D) and/or two-dimensional (2D) results by the tensor-product technique. When the element is not tensor-product due to either geometry or construction, the problem becomes tricky. In [6] Brandts and Křížek surveyed techniques and typical difficulties of superconvergence problems in 3D FEs. Three different methods were reviewed, namely, the *element orthogonality analysis* introduced by Chen [9], Zhu and Lin [26], et al.; the *symmetry theory* due to Schatz, Sloan, and Wahlbin [19]; and *computer-based methods* proposed by Babuška et al. [5].

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Following the reasoning of computer-based methods, Zhang [23, 24] proposed an analytic approach to locate derivative superconvergence points for some 2D rectangular and 3D hexahedral elements. In this paper, we systematically extend this analytic approach to identify superconvergence points for 3D higher order FEs of different geometries, including pentahedral (triangular prism) and tetrahedral elements. Specifically, Lagrangian and Serendipity elements of both a hexahedron and a pentahedron are considered; two patterns of tetrahedral elements are also investigated. Superconvergence results for function values and derivatives are provided for elements up to certain degrees. The reader is referred to [25] as a supplement for computational aspects. To the best of our knowledge, many results in this paper are reported for the first time in the literature.

The article is organized as follows. Section 2 contains some preliminaries. Hierarchic shape functions for the elements considered in this paper are constructed in section 3. In section 4, we introduce a decomposition of the FE spaces, which provides us a way to specify the polynomial spaces needed in locating superconvergence points. The structures of the spaces are illustrated in section 5. Section 6 is devoted to presenting the main superconvergence results in different elements. Some concluding remarks are made in the last section.

2. Preliminaries.

2.1. Master cell and reference cell. Consider superconvergence near $\boldsymbol{x}^0 \in \Omega$. We assume that the local FE triangulation is based on a uniform hexahedral mesh (which may be further decomposed into pentahedral or tetrahedral elements) near $\boldsymbol{x}^0$. Without loss of generality, we may assume that the point $\boldsymbol{x}^0$ is the center of one cell in the mesh, which is referred to as the *master cell*. Denote a cell centered at $\boldsymbol{x} = (x_1, x_2, x_3) \in \Omega$ with side $2l$ by

$$c(\boldsymbol{x}, l) = \{\boldsymbol{y} = (y_1, y_2, y_3) \in \Omega \mid |y_i - x_i| \le l,\ i = 1, 2, 3\}.$$

The master cell is hence $c(\boldsymbol{x}^0, h)$, where h is the *mesh parameter*. Let $\Omega_1 = c(\boldsymbol{x}^0, H)$ and $\Omega_0 = c(\boldsymbol{x}^0, 2H) \subset\subset \Omega$ with $H = h^\delta$, $0 < 3\delta/2 < 1$.

Using the reference element technique, the study in the master cell $c(\boldsymbol{x}^0, h)$ can be equivalently conducted in the *reference cell* $K = [-1, 1]^3$. In this context, we will use (x_1, x_2, x_3) for the standard Euclidean coordinates in the master cell (or physical domain Ω), and use (ξ, η, ζ) for those in the reference cell.

2.2. Function spaces. In this paper, we use standard notation for Lebesgue and Sobolev spaces and the corresponding norms.

We denote by Π_n the space of polynomials in three variables with degree not exceeding n. Associated with a particular partition of the reference cell K, we denote by $\Pi_n^w(K)$ ($\Pi_n^\pi(K)$, respectively) the set of *piecewise continuous* (*periodic piecewise continuous*, respectively) polynomials of degree not greater than n. Here, we say a function f is *periodic with respect to* K, or simply *periodic*, if $f(1, \eta, \zeta) = f(-1, \eta, \zeta)$, $f(\xi, 1, \zeta) = f(\xi, -1, \zeta)$, and $f(\xi, \eta, 1) = f(\xi, \eta, -1)$ for any $\xi, \eta, \zeta \in [-1, 1]$. Functions *periodic with respect to the master cell* can be analogously defined. These functions can be periodically extended to Ω_0, which are referred to as $2h$-*periodic functions*.

Let $\mathcal{H}_{n+1}$ be the space of *homogeneous harmonic polynomials* in three variables of degree $n + 1$. For $\boldsymbol{x} \in \mathbb{R}^3$, denote its Euclidean norm by $|\boldsymbol{x}|$. It is proved in [3, p. 76] that every $p \in \Pi_{n+1} \backslash \Pi_n$ can be expressed as $p = p_1 + |\boldsymbol{x}|^2 p_2$ with $p_1 \in \mathcal{H}_{n+1}$ and $p_2 \in \Pi_{n-1} \backslash \Pi_{n-2}$. We have the following proposition [3].

PROPOSITION 2.1. *For* $n \ge 1$, *the space* $\mathcal{H}_{n+1}$ *has dimension* $\dim \mathcal{H}_{n+1} = 2n+3$.

Furthermore, with help of the *Kelvin transform*, we can find an explicit basis of $\mathcal{H}_{n+1}$. For the detailed approach, see [3, p. 92].

We denote by V_n the polynomial FE space of degree n, and by V_n^π the collection of $2h$-periodic functions in V_n. We denote also by $V_n(K)$ and $V_n^\pi(K)$ the corresponding FE spaces on K.

2.3. Auxiliary functions. The following auxiliary functions will be used later. Let P_k be the *Legendre polynomial* of degree k on $[-1,1]$. Define

$$\phi_k(x) = \int_{-1}^{x} P_{k-1}(t)\ \mathrm{d}t, \quad k = 2, 3, \dots. \tag{2.1}$$

We define also

$$\begin{aligned} \tilde{\phi}_2 &= \phi_2 + \frac{1}{3}, \\ \tilde{\phi}_k &= \phi_k \quad \forall k = 3, 4, \dots. \end{aligned} \tag{2.2}$$

Note that ϕ_k and $\tilde{\phi}_k$ are polynomials of degree k for $k \geq 2$.

2.4. Residue estimates. Let u be the weak solution of (1.1), and u_h be the FE approximation of u in V_n. It follows that

$$a(u - u_h, v) = 0 \qquad \forall v \in V_n^{comp}, \tag{2.3}$$

where V_n^{comp} is the collection of functions in V_n with compact support in Ω, and the bilinear form $a(\cdot,\cdot)$ is defined as

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \ \mathrm{d}\Omega.$$

Under various conditions given in [20], the following L_∞ estimates of $u - u_h$ and its derivatives are obtained.

LEMMA 2.2. *Let u and u_h be the weak solution of* (1.1) *and its FE approximation, respectively. Assume that $s \geq 0$ and $1 \leq q \leq \infty$. Then there exists a constant C independent of u, u_h, h, H, and $\boldsymbol{x}^0$ such that*

$$\|u - u_h\|_{L_\infty(\Omega_1)} \leq C \left(\ln \frac{H}{h}\right)^{\bar{n}} \min_{v \in V_n} \|u - v\|_{L_\infty(\Omega_0)} + \ CH^{-s-3/q} \|u - u_h\|_{W_q^{-s}(\Omega_0)}.$$

Here $\bar{n} = 1$ if $n = 1$, and $\bar{n} = 0$ otherwise.

LEMMA 2.3. *Under the same hypotheses as in Lemma* 2.2, *there exists a constant C independent of u, u_h, h, H, and $\boldsymbol{x}^0$ such that*

$$\begin{aligned} \|u - u_h\|_{W_\infty^1(\Omega_1)} \leq{} & C \min_{v \in V_n} \left(|u - v|_{W_\infty^1(\Omega_0)} + H^{-1} \|u - v\|_{L_\infty(\Omega_0)} \right) \\ & + \ CH^{-1-s-\frac{3}{q}} \|u - u_h\|_{W_q^{-s}(\Omega_0)}. \end{aligned}$$

The proofs of Lemmas 2.2 and 2.3 can be found in [20].

For $n \geq 1$, we assume that

$$\|u - u_h\|_{L_\infty(\Omega_0)} \leq C h^{n+1-L}, \tag{2.4}$$

with $L \geq 0$ and $L + 3\delta/2 < 1$. This assumption implies that pollution effects from outside of the domain Ω_0 have been properly controlled and the error loss is of order h^L. Moreover, for $n > 1$, we assume that

$$\|u - u_h\|_{W_\infty^{-1}(\Omega_0)} \leq Ch^{n+2-\Lambda}, \tag{2.5}$$

with $\Lambda \geq 0$ and $\Lambda + \delta < 1$.

The following theorems are analogous to the 2D results in [5] and [18]. We refer the reader also to [22].

THEOREM 2.4. *Assume that the FE mesh over Ω is locally translation invariant on Ω_0 and (2.4) holds. Then, for $n \geq 1$, we have*

$$\frac{\partial}{\partial x_i}(u - u_h)(\boldsymbol{x}) = \frac{\partial \psi}{\partial x_i}(\boldsymbol{x}) + R_{x_i}(\boldsymbol{x}), \qquad i = 1, 2, 3, \quad \boldsymbol{x} \in \Omega_1,$$

where

$$\|R_{x_i}\|_{L_\infty(\Omega_1)} \leq Ch^n \left(h^\delta + h^{1-L-\frac{3}{2}\delta}\right).$$

Here, ψ is the periodic extension of a piecewisely defined polynomial of degree $n+1$, which is periodic with respect to the master cell.

Proof. Let Q be the $n+1$st order Taylor expansion of u at $\boldsymbol{x}_0$. By approximation theory one has

$$\|u - Q\|_{W_\infty^s(\Omega_0)} \leq CH^{n+2-s}, \qquad 0 \leq s \leq n+2. \tag{2.6}$$

Let $N_h Q$ be the *Neumann projection* of Q into $V_n(\Omega_0)$ in the sense that

$$a(Q - N_h Q, v) = 0 \quad \forall v \in V_n(\Omega_0), \quad \text{and} \quad \int_{\Omega_0} (Q - N_h Q)\, \mathrm{d}\Omega = 0. \tag{2.7}$$

Let $R_Q = (u - u_h) - (Q - N_h Q)$. From (2.3) and (2.7) one has

$$a(R_Q, v) = 0 \qquad \forall v \in V_n^{comp}(\Omega_0).$$

By Lemma 2.3 with $s = 0$ and $q = 2$, it follows that, for $i = 1, 2$, and 3,

$$\begin{aligned} \left\|\frac{\partial R_Q}{\partial x_i}\right\|_{L_\infty(\Omega_1)} \leq C & \min_{v \in V_n(\Omega_0)} |(u - Q) - v|_{W_\infty^1(\Omega_0)} \\ & + H^{-1}\|(u - Q) - v\|_{L_\infty(\Omega_0)}) + CH^{-5/2}\|R_Q\|_{L_2(\Omega_0)}. \end{aligned} \tag{2.8}$$

Let $v \in V_n(\Omega_0)$ be the interpolation of $u - Q$. By approximation theory and (2.6) with $s = n + 1$,

$$|(u - Q) - v|_{W_\infty^1(\Omega_0)} \leq Ch^n \|u - Q\|_{W_\infty^{n+1}(\Omega_0)} \leq Ch^n H. \tag{2.9}$$

Similarly,

$$H^{-1}\|(u - Q) - v\|_{L_\infty(\Omega_0)} \leq Ch^{n+1}. \tag{2.10}$$

Recalling (2.4), one has

$$\|u - u_h\|_{L_2(\Omega_0)} \leq CH\|u - u_h\|_{L_\infty(\Omega_0)} \leq CHh^{n+1-L}. \tag{2.11}$$

It follows, by the standard L_2 estimate, that

$$(2.12)\quad \begin{aligned}\|Q - N_hQ\|_{L_2(\Omega_0)} &\le Ch^{n+1}\|Q\|_{W_2^{n+1}(\Omega_0)}\\ &\le Ch^{n+1}H^{3/2}\|Q\|_{W_\infty^{n+1}(\Omega_0)} \le CH^{3/2}h^{n+1-L}.\end{aligned}$$

Hence, from (2.8)–(2.12), one gets

$$(2.13)\quad \left\|\frac{\partial R_Q}{\partial x_i}\right\|_{L_\infty(\Omega_1)} \le Ch^n(H + H^{-3/2}h^{1-L} + H^{-1}h^{1-L}).$$

Let $\rho = Q - I_hQ$, where I_hQ is an interpolation of Q in $V_n(\Omega_0)$. Since, as indicated in [5], I_hQ is translation invariant by $2h$, it can be shown that ρ is $2h$-periodic; cf. [5, 22]. Set $\psi = \rho - N_h^\pi\rho$, where $N_h^\pi\rho$ is the Neumann projection of ρ into $V_n^\pi(\Omega_0)$; i.e., $N_h^\pi\rho$ is the $2h$-periodic version of $N_h\rho$. It follows immediately from (2.7) that

$$(2.14)\quad a(\psi, v) = 0 \qquad \forall v \in V_n^{comp}(\Omega_0).$$

Rewrite $Q - N_hQ = \psi + [Q - N_hQ - \psi]$. From (2.7) and (2.14), one gets

$$a(Q - N_hQ - \psi, v) = 0 \qquad \forall v \in V_n^{comp}(\Omega_0).$$

Notice that $v = Q - N_hQ - \psi = I_hQ + N_h^\pi\rho - N_hQ \in V_n(\Omega_0)$. It follows from Lemma 2.3 with $s = 0, q = 2$ that

$$(2.15)\quad \begin{aligned}\|Q - N_hQ - \psi\|_{W_\infty^1(\Omega_1)} &\le CH^{-5/2}\|Q - N_hQ - \psi\|_{L_2(\Omega_0)}\\ &\le CH^{-5/2}\left(\|Q - N_hQ\|_{L_2(\Omega_0)} + \|\psi\|_{L_2(\Omega_0)}\right).\end{aligned}$$

Another application of the L_2 estimate gives

$$(2.16)\quad \|Q - N_hQ\|_{L_2(\Omega_0)} \le Ch^{n+1-L}H^{3/2}.$$

By a duality argument (see, e.g., [8]),

$$(2.17)\quad \begin{aligned}\|\psi\|_{L_2(\Omega_0)} &\le Ch\|\rho\|_{W_2^1(\Omega_0)} \le Ch^{n+1}\|Q\|_{W_2^{n+1}(\Omega_0)}\\ &\le Ch^{n+1}H^{3/2}\|Q\|_{W_\infty^{n+1}(\Omega_0)} \le Ch^{n+1-L}H^{3/2}.\end{aligned}$$

From (2.15)–(2.17), one concludes that

$$(2.18)\quad \|Q - N_hQ - \psi\|_{W_\infty^1(\Omega_1)} \le Ch^{n+1-L}H^{-1}.$$

Finally, set $R = (u - u_h) - \psi$ and $R_{x_i} = \partial R/\partial x_i$, $i = 1, 2, 3$. From (2.13) and (2.18), one gets

$$\|R_{x_i}\|_{L_\infty(\Omega_1)} \le Ch^n(H + H^{-3/2}h^{1-L} + H^{-1}h^{1-L}).$$

Note that $H = h^\delta < 1$, and the desired result follows. □

THEOREM 2.5. *Assume that the FE mesh on Ω is locally translation invariant over Ω_0. Assume also that* (2.5) *holds. Then, for $n > 1$, we have*

$$(u - u_h)(\boldsymbol{x}) = \psi(\boldsymbol{x}) + R(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_1,$$

where

$$\|R\|_{L_\infty(\Omega_1)} \le Ch^{n+1}\left(h^\delta + h^{1-\Lambda-\delta}\right),$$

and ψ *is as described in Theorem* 2.4.

Proof. Define N_hQ, R_Q, and R as in Theorem 2.4. By Lemma 2.2 with $q = \infty, s = 1$,

$$\|R_Q\|_{L_\infty(\Omega_1)} \le C \min_{v\in V_n(\Omega_0)} \|(u-Q)-v\|_{L_\infty(\Omega_0)} + CH^{-1}\|R_Q\|_{W_\infty^{-1}(\Omega_0)}. \tag{2.19}$$

Let $v \in V_n(\Omega_0)$ be the interpolation of $u - Q$. From (2.6) with $s = n+1$, one gets

$$\|(u-Q)-v\|_{L_\infty(\Omega_0)} \le Ch^{n+1}\|u-Q\|_{W_\infty^{n+1}(\Omega_0)} \le Ch^{n+1}H. \tag{2.20}$$

By assumption (2.5)

$$\|R_Q\|_{W_\infty^{-1}(\Omega_0)} \le \|u-u_h\|_{W_\infty^{-1}(\Omega_0)} + \|Q-N_hQ\|_{W_\infty^{-1}(\Omega_0)} \le Ch^{n+2-\Lambda}. \tag{2.21}$$

Combining (2.19)–(2.21) yields

$$\|R_Q\|_{L_\infty(\Omega_1)} \le Ch^{n+1}H + CH^{-1}h^{n+2-\Lambda}. \tag{2.22}$$

Since $Q - N_hQ - \psi \in V_n(\Omega_0)$, from Lemma 2.2,

$$\begin{aligned}\|Q-N_hQ-\psi\|_{L_\infty(\Omega_1)} &\le CH^{-1}\|Q-N_hQ-\psi\|_{W_\infty^{-1}(\Omega_0)}\\ &\le CH^{-1}\left(\|Q-N_hQ\|_{W_\infty^{-1}(\Omega_0)} + \|\psi\|_{W_\infty^{-1}(\Omega_0)}\right).\end{aligned} \tag{2.23}$$

By assumption (2.5)

$$\|Q-N_hQ\|_{W_\infty^{-1}(\Omega_0)} \le Ch^{n+2-\Lambda}. \tag{2.24}$$

By a duality argument,

$$\|\psi\|_{W_\infty^{-1}(\Omega_0)} \le \|\rho\|_{W_\infty^{-1}(\Omega_0)} \le Ch^{n+2}\|Q\|_{W_\infty^{n+1}(\Omega_0)} \le Ch^{n+2-\Lambda}. \tag{2.25}$$

Therefore, (2.23)–(2.25) give

$$\|Q-N_hQ-\psi\|_{L_\infty(\Omega_1)} \le CH^{-1}h^{n+2-\Lambda}. \tag{2.26}$$

From (2.22) and (2.26), one gets

$$\|R\|_{L_\infty(\Omega_1)} \le Ch^{n+1}(H + H^{-1}h^{1-\Lambda}), \tag{2.27}$$

which completes the proof of the theorem. □

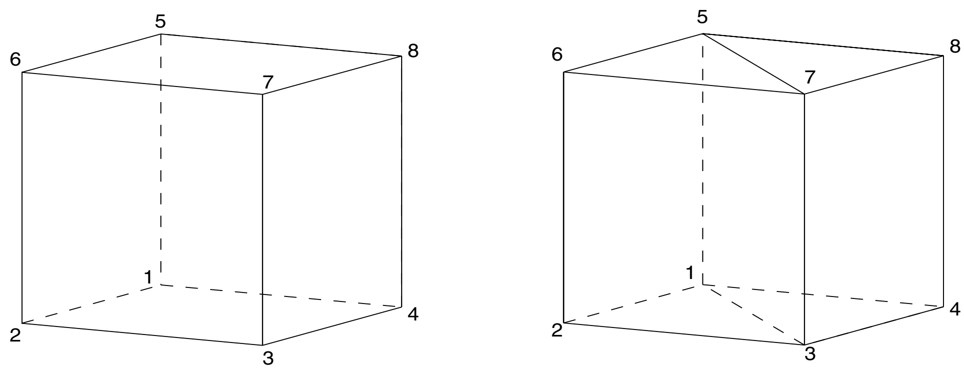
Remark 2.6. Theorem 2.5 may fail when $n = 1$. In fact, even for the one-dimensional case, there is no superconvergence point for function values when piecewise linear elements are employed [22].

Remark 2.7. Notice that ψ is a piecewise polynomial on the master cell and is $2h$-periodic in Ω_0. Theorems 2.4 and 2.5 tell us that the FE local errors in derivatives and function values can be majorized by $\partial\psi/\partial x_i$ and ψ, respectively. Therefore, the task of finding superconvergence points in Ω_1 can be narrowed down to the master cell or, equivalently, to the reference cell under polynomial inputs. The study of superconvergence in a locally translation invariant mesh is from now on carried out in K.

3. Three-dimensional FEs. In this section, hexahedral, pentahedral, and tetrahedral elements are considered. In a hexahedral mesh, a cell in the physical domain serves as an element, which can be mapped to the reference cell K illustrated in Figure 1(a) by the affine mapping

$$(\xi, \eta, \zeta) = \left(\frac{x_1 - x_1^*}{h}, \frac{x_2 - x_2^*}{h}, \frac{x_3 - x_3^*}{h} \right), \tag{3.1}$$

where $\boldsymbol{x}^* = (x_1^*, x_2^*, x_3^*)$ is the center of the physical cell.



(a) Hexahedral Element (b) Pentahedral Elements

FIG. 1. *Partition of K for hexahedral and pentahedral elements.*

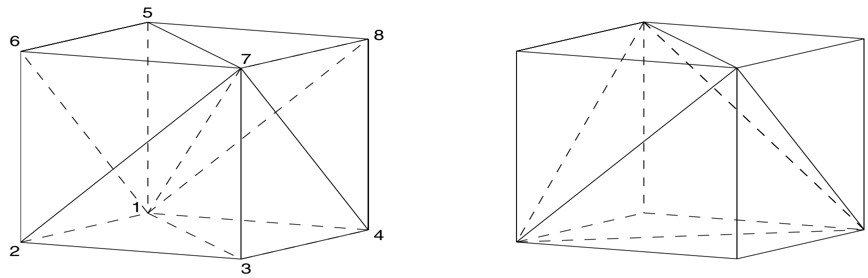
For pentahedral elements, our consideration is for a mesh in which each cell in the physical domain is mapped to the reference cell illustrated in Figure 1(b) by (3.1). The cross-section of this 3D mesh in the x_1x_2-plane is a 2D triangular mesh in a regular pattern [18].

For tetrahedral meshes, two schemes are usually used to partition a unit cell. See Figure 2. Scheme 1 is called *Kuhn partition* of the unit cell, which produces six isomorphic tetrahedra. The master cell can be mapped to the reference cell K by the mapping (3.1). Superconvergence results in [7, 9, 12, 13] are all for tetrahedral elements under Kuhn partition.

Scheme 2 divides a cell into five tetrahedra, four of which are isomorphic. Unlike the previous partitions, to get a translation invariant patch under this scheme, at least eight cubes are needed. As illustrated in Figure 3, there are two candidates for a master cell, which are essentially the same. We consider the first patch in this paper. See Figure 4 for partition of the reference cell. Few superconvergence results are available in the literature for tetrahedral elements under this partition.

We next construct FE spaces $V_n(K)$ and $V_n^\pi(K)$. Hierarchic basis functions are used for $V_n(K)$, which are organized into four categories: nodal shape functions, edge modes, face modes, and internal modes [21]. The nodal shape functions are the standard shape functions, each of which has value 1 at the corresponding node and value 0 at the other nodes. The modal shape functions are, in principle, products of the nodal shape functions of the nodes in the edges, faces, or polyhedra.

Periodic bases of $V_n^\pi(K)$ can be obtained from the basis of $V_n(K)$. In particular, the sum of the nodal shape functions is periodic. The sums of the same order edge modes associated with the three groups of parallel sides, and the sums of the same



Scheme 1 Scheme 2

FIG. 2. *Schemes partitioning unit cell into tetrahedra.*

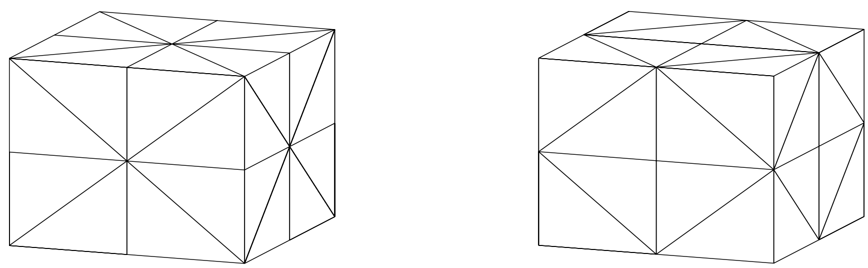
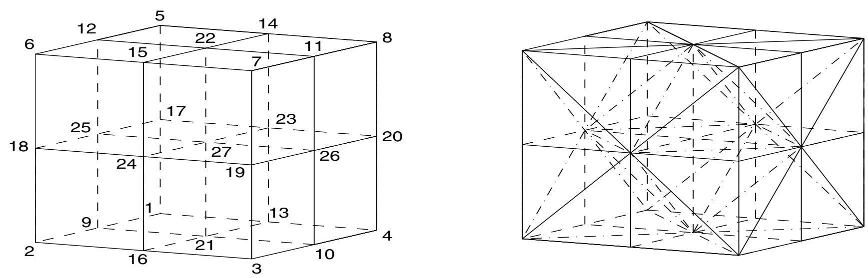


FIG. 3. *Translation invariant patches under scheme* 2.



Nodes in K Partition of K

FIG. 4. *Partition of* K *for tetrahedral elements under scheme* 2.

order face modes for the opposite faces, are all periodic functions. The internal modes are automatically periodic. For more details, see [17].

A basis of Lagrangian pentahedral FE space and a basis of Serendipity pentahedral FE space of degree n are constructed as an example. In Figure 1(b), $\{(\xi, \eta, \zeta) | \xi \geq \eta\}$ is referred to as the first element and the other the second element. We consider Lagrangian elements first.

(1) *Nodal shape functions*. There are 8 nodal shape functions, which are defined

as

$$N^p(\xi,\eta,\zeta) = \begin{cases} \frac{1}{4}(1-\xi)(1+\zeta_p\zeta), \\ \frac{1}{4}(1-\eta)(1+\zeta_p\zeta), \end{cases} \quad p = 1,5,$$

$$N^p(\xi,\eta,\zeta) = \begin{cases} \frac{1}{4}(\xi-\eta)(1+\zeta_p\zeta), \\ 0, \end{cases} \quad p = 2,6,$$

$$N^p(\xi,\eta,\zeta) = \begin{cases} \frac{1}{4}(1+\eta)(1+\zeta_p\zeta), \\ \frac{1}{4}(1+\xi)(1+\zeta_p\zeta), \end{cases} \quad p = 3,7,$$

$$N^p(\xi,\eta,\zeta) = \begin{cases} 0, \\ \frac{1}{4}(\eta-\xi)(1+\zeta_p\zeta), \end{cases} \quad p = 4,8,$$

where ζ_p is the ζ-coordinate of the pth node. Hereafter, the two expressions of each basis function are its definitions in elements one and two, respectively.

(2) *Edge modes*. There are $(n-1)$ edge modes on each side with index $i = 2, \dots, n$. In particular, the edge modes on $l_{1,2}$ are defined as

$$E_i^{1,2} = \begin{cases} \frac{1}{8}(1-\xi)(\xi-\eta)(1-\zeta)\varphi_{i-2}(\xi - \frac{1}{2}\eta - \frac{1}{2}), \\ 0, \end{cases}$$

where φ_i's are defined from the auxiliary functions in (2.1) by

$$\phi_{i+2}(x) = \frac{1}{4}(1-x^2)\varphi_i(x).$$

Note that each $\phi_k(x)$ has a factor $1-x^2$. Analogously, we define basis functions on sides parallel to the ξ and η axes such as $E_i^{1,4}, E_i^{5,6}, E_i^{5,8}, E_i^{2,3}, E_i^{3,4}, E_i^{6,7}, E_i^{7,8}$. On edges $l_{1,5}$ and $l_{2,6}$, we have

$$E_i^{1,5} = \begin{cases} \frac{1}{8}(1-\xi)(1-\zeta^2)\varphi_{i-2}(\zeta), \\ \frac{1}{8}(1-\eta)(1-\zeta^2)\varphi_{i-2}(\zeta), \end{cases}$$

$$E_i^{2,6} = \begin{cases} \frac{1}{8}(\xi-\eta)(1-\zeta^2)\varphi_{i-2}(\zeta), \\ 0. \end{cases}$$

The edge modes associated with $l_{3,7}$ and $l_{4,8}$ are similarly defined. Finally, for the diagonal edge $l_{1,3}$, we define

$$E_i^{1,3} = \begin{cases} \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)\varphi_{i-2}(\frac{\xi+\eta}{2}), \\ \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)\varphi_{i-2}(\frac{\xi+\eta}{2}), \end{cases}$$

and we define $E_i^{5,7}$ similarly.

(3) *Face modes*. On each rectangular face, $(n-1)^2$ shape functions are defined when $n \geq 2$. For instance, on faces 2-3-6-7 and 1-3-5-7, the face modes are

$$F_{i,j}^{2,3,6,7} = \begin{cases} \frac{1}{16}(1+\eta)(\xi-\eta)(1-\zeta^2)\varphi_{i-2}(\eta - \frac{\xi}{2} + \frac{1}{2})\varphi_{j-2}(\zeta), \\ 0, \end{cases}$$

$$F_{i,j}^{1,3,5,7} = \begin{cases} \frac{1}{16}(1-\xi)(1+\eta)(1-\zeta^2)\varphi_{i-2}(\frac{\xi+\eta}{2})\varphi_{j-2}(\zeta), \\ \frac{1}{16}(1+\xi)(1-\eta)(1-\zeta^2)\varphi_{i-2}(\frac{\xi+\eta}{2})\varphi_{j-2}(\zeta), \end{cases}$$

with $i, j = 2, \ldots, n$. On each triangular face, $(n-1)(n-2)/2$ shape functions are defined when $n \geq 3$. For example, on face 1-2-3, the face modes are

$$F_{i,j}^{1,2,3} = \begin{cases} \frac{1}{16}(1-\xi)(\xi-\eta)(1+\eta)(1-\zeta)\xi^i\eta^j, \\ 0, \end{cases}$$

with $i, j = 0, \ldots, n-3$ and $i + j = 0, \ldots, n-3$.

(4) *Internal modes*. There are $(n-1)^2(n-2)/2$ internal modes in every element. For $i, j = 0, \ldots, n-3$, $i+j = 0, \ldots, n-3$, and $k = 2, \ldots, n$, we define internal modes of the first element as

$$I_{i,j,k}^{e_1} = \begin{cases} \frac{1}{32}(1-\xi)(\xi-\eta)(1+\eta)(1-\zeta^2)\xi^i\eta^j\varphi_{k-2}(\zeta), \\ 0. \end{cases}$$

Internal modes of the second elements are analogous.

For Serendipity elements, the nodal shape functions, edge modes, and triangular face modes are congruent as above. However, rectangular face modes are defined as above only when $n \geq 4$, with $i, j = 2, \ldots, n-2$ and $i + j = 4, \ldots, n$. Internal modes are involved when $n \geq 5$; these are defined as above with $i, j = 0, \ldots, n-5$, $k = 2, \ldots, n-3$, and $i + j + k = 2, \ldots, n-3$.

For hierarchic basis functions of hexahedral and tetrahedral elements, the reader is referred to [4, 17, 25].

4. Periodic FE approximation. Let $\Phi_{n+1}(K)$ be a subset of $\Pi_{n+1}^{\pi}(K)$, so that each function ψ in $\Phi_{n+1}(K)$ can be decomposed into the polynomial space $\Pi_{n+1}(K)$ and the piecewise continuous polynomial space $\Pi_n^w(K)$, and

$$\int_K \nabla\psi \cdot \nabla v = 0 \quad \forall v \in \Pi_n^{\pi}(K) \quad \text{and} \quad \int_K \psi = 0. \tag{4.1}$$

That is, for any $\psi \in \Phi_{n+1}(K)$, we have $\psi = \chi + r$ satisfying (4.1), where $\chi \in \Pi_{n+1}(K)$ and $r \in \Pi_n^w(K)$. It is straightforward to show the following result [17, 18].

PROPOSITION 4.1. $\dim \Phi_{n+1}(K) = \dim \Pi_{n+1}(K) - \dim \Pi_n(K)$.

The proposition says that, for each monomial of degree $n+1$, there is a unique function in $\Phi_{n+1}(K)$ corresponding to it. Similarly, for each homogeneous harmonic polynomial of degree $n+1$, there is a unique function in $\Phi_{n+1}(K)$ corresponding to it. In particular, we define the following set, which will be used to study superconvergence of FE solutions for Laplace equations:

$$\Phi_{n+1}^{\mathcal{H}}(K) = \{\, \psi \in \Phi_{n+1}(K) \mid \psi = \chi + r,\ \chi \in \mathcal{H}_{n+1}(K),\ r \in \Pi_n^w(K) \,\}. \tag{4.2}$$

We next introduce a process for determining a basis for $\Phi_{n+1}(K)$ and $\Phi_{n+1}^{\mathcal{H}}(K)$. Here we use tetrahedral elements under scheme 1 as an example; see Figure 2. Let u be a polynomial of degree $n+1$ and define an interpolation operator I_n into $V_n(K)$, such that the following hold:

(1) $I_n u(\xi_p, \eta_p, \zeta_p) = u(\xi_p, \eta_p, \zeta_p)$, $p = 1, \ldots, 8$, where (ξ_p, η_p, ζ_p) is the pth node of K.

(2) Along each edge l in the partition of K,

$$\int_l (u - I_n u) r^j \, \mathrm{d}r = 0, \quad j = 0, 1, \ldots, n-2. \tag{4.3}$$

(3) On each face S in the partition of K,

$$\int_S (u - I_n u) r^j s^k \, \mathrm{d}r \, \mathrm{d}s = 0, \quad j, k \geq 0, \quad j + k = 0, 1, \ldots, n - 3. \tag{4.4}$$

(4) In each tetrahedron T in the partition of K,

$$\int_T (u - I_n u) \xi^j \eta^k \zeta^l \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta = 0, \quad j, k, l \geq 0, \quad j + k + l = 0, 1, \ldots, n - 4. \tag{4.5}$$

Note that $u - I_n u$ is a periodic function in K [4, 22]. Thus, we may find a periodic FE approximation z^π of $u - I_n u$ by solving

$$\int_K \nabla(u - I_n u - z^\pi) \cdot \nabla v = 0 \quad \forall v \in V_n^\pi(K). \tag{4.6}$$

To determine z^π uniquely, we also need

$$\int_K (u - I_n u - z^\pi) = 0. \tag{4.7}$$

Then $\psi|_u = u - I_n u - z^\pi \in \Phi_{n+1}(K)$. See [17, 25] for examples and more details. Letting u run through all monomials of degree $n + 1$, we obtain a basis of space $\Phi_{n+1}(K)$ by this approach. Using the process described above, we can also have a unique correspondence of $u \in \mathcal{H}_{n+1}$ in $\Phi_{n+1}^{\mathcal{H}}(K)$.

With a symbolic computation software like Maple, the basis functions of $\Phi_{n+1}(K)$ and $\Phi_{n+1}^{\mathcal{H}}(K)$ can be explicitly determined. Therefore, the process is analytic. Here we list only results for tetrahedral elements of scheme 1. As for results in other geometric shapes, such as tetrahedral elements of scheme 2 and pentahedral elements, the reader is referred to [17].

Notice that the process developed above provides functions ψ which satisfy conditions in Theorems 2.4 and 2.5. The following theorems are thus consequences of Theorems 2.4 and 2.5.

THEOREM 4.2. *Let the assumptions of Theorems* 2.4 *and* 2.5 *hold. Then the following hold:*

(i) *For $n > 1$, the function value superconvergence points of $V_n(K)$ for the Poisson equation are the intersections of the following collection of curves:*

$$\{ \psi = 0 \mid \psi \in \Phi_{n+1}(K) \backslash V_n(K) \} .$$

(ii) *For $n \geq 1$, the derivative superconvergence points of $V_n(K)$ in the ξ-direction for the Poisson equation are the intersections of the following collection of curves:*

$$\left\{ \frac{\partial \psi}{\partial \xi} = 0 \mid \psi \in \Phi_{n+1}(K) \backslash V_n(K) \right\} .$$

A similar result holds for derivatives in other directions.

THEOREM 4.3. *Let the assumptions of Theorems* 2.4 *and* 2.5 *hold. Then the following hold:*

(i) *For $n > 1$, the function value superconvergence points of $V_n(K)$ for the Laplace equation are the intersections of the following collection of curves:*

$$\{ \psi = 0 \mid \psi \in \Phi_{n+1}^{\mathcal{H}}(K) \backslash V_n(K) \} .$$

(ii) *For $n \geq 1$, the derivative superconvergence points of $V_n(K)$ in the ξ-direction for the Laplace equation are the intersections of the following collection of curves:*

$$\left\{ \frac{\partial \psi}{\partial \xi} = 0 \mid \psi \in \Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K) \right\}.$$

A similar result holds for derivatives in other directions.

Remark 4.4. From Theorems 4.2 and 4.3, one concludes that, if bases of $\Phi_{n+1}(K)\backslash V_n(K)$ and $\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)$ are determined, then the superconvergence points can be located by solving a system of polynomial equations, which is a routine exercise of numerical computation.

5. Structures of $\boldsymbol{\Phi_{n+1}(K)\backslash V_n(K)}$ and $\boldsymbol{\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)}$. To determine the spaces $\Phi_{n+1}(K)\backslash V_n(K)$ and $\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)$, one may first determine a basis of $\Pi_{n+1}(K) \setminus V_n(K)$ and $\mathcal{H}_{n+1}(K) \setminus V_n(K)$, respectively, and then use the approach developed in section 4 to find the corresponding basis functions in the desired spaces.

5.1. Hexahedral elements. Consider Lagrangian hexahedral elements. We note that the nth degree local FE space $V_n(K)$ contains not only functions in $\Pi_n(K)$, but also polynomials of degree higher than n. In particular,

$$\Pi_{n+1}(K) \setminus V_n(K) = \text{Span}\left\{\xi^{n+1}, \eta^{n+1}, \zeta^{n+1}\right\}.$$

It is straightforward to verify that

$$\Phi_{n+1}(K)\backslash V_n(K) = \text{Span}\left\{\tilde{\phi}_{n+1}(\xi), \tilde{\phi}_{n+1}(\eta), \tilde{\phi}_{n+1}(\zeta)\right\},$$

where $\tilde{\phi}_n$ is defined in (2.2). $\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)$ can be specified analogously.

For Serendipity hexahedral elements, there are fewer basis functions in $V_n(K)$. In particular, we have

$$\Pi_{n+1}(K) \setminus V_n(K) = \text{Span}\left\{\xi^i \eta^j \zeta^k \mid i+j+k = n+1,\ i,j,k \neq 1\right\}.$$

Then $\Phi_{n+1}(K)\backslash V_n(K)$ is determined accordingly, and so is $\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)$. See [23] for similar formulations obtained from a different approach.

5.2. Pentahedral elements. Consider Lagrangian pentahedral elements. We have

$$\Pi_{n+1}(K) \setminus V_n(K) = \text{Span}\left\{\zeta^{n+1}, \xi^{n+1-i}\eta^i \mid i = 0, \ldots, n+1\right\}.$$

Then $\Phi_{n+1}(K)\backslash V_n(K)$ can be determined. It turns out to be spanned by $\tilde{\phi}_{n+1}(\zeta)$ and the basis of the triangular elements for a regular pattern in [18]. In addition, $\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)$ has a basis consisting of $\tilde{\phi}_{n+1}(\zeta)$ and two harmonic basis functions of triangular regular pattern in [18].

For Serendipity pentahedral elements, we have

$$\Pi_{n+1}(K) \setminus V_n(K) = \text{Span}\left\{\xi^{n+1-i-j}\eta^i\zeta^j \mid i, j, i+j = 0, \ldots, n+1,\ j \neq 1, n\right\}.$$

Using the process introduced in section 4, we can determine $\Phi_{n+1}(K)\backslash V_n(K)$ and $\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)$ accordingly. Note that when $n = 1$ and 2, the spaces for Serendipity elements are the same as those for Lagrangian elements. When $n \geq 3$, the spaces are different.

TABLE 1
Function value superconvergence results of hexahedral FE.

	Lagrangian hexahedral FE	Serendipity hexahedral FE	
n	Poisson & Laplace	Poisson	Laplace
2	HF_2	SPH	SPH
3	HF_3	None	HSet1
4	HF_4	SPH	SPH
5	HF_5	None	None
6	HF_6	SPH	SPH

TABLE 2
ξ-derivative superconvergence results of hexahedral FE.

	Lagrangian hexahedral FE	Serendipity hexahedral FE	
n	Poisson & Laplace	Poisson	Laplace
1	HD_1	HD_1	HD_1
2	HD_2	HD_2	HD_2
3	HD_3	HSet2	HSet4
4	HD_4	None	None
5	HD_5	HSet3	HSet3
6	HD_6	None	None

5.3. Tetrahedral elements. For tetrahedral elements, we have

$$\Pi_n(K) \subset \Pi_n^w(K) = V_n(K).$$

Note that $V_n(K)$ does not contain any polynomial of degree greater than n; thus,

$$\Pi_{n+1}(K) \setminus V_n(K) = \Pi_{n+1}(K) \setminus \Pi_n(K).$$

Therefore, $\Phi_{n+1}(K)\backslash V_n(K) = \Phi_{n+1}(K)$. A basis of $\Phi_{n+1}^{\mathcal{H}}(K)$ can be obtained from a basis of $\mathcal{H}_{n+1}$.

6. Superconvergence results. Once the bases of spaces $\Phi_{n+1}(K)\backslash V_n(K)$ and $\Phi_{n+1}^{\mathcal{H}}(K)\backslash V_n(K)$ are determined, one may apply Theorems 4.2 and 4.3 to the basis functions and obtain superconvergence points for the Poisson and Laplace equations, respectively.

6.1. Hexahedral elements. In Tables 1 and 2, we collect superconvergence results for Lagrangian and Serendipity hexahedral elements of degree up to 6, though our method can produce superconvergence results for elements of higher degrees. Here, "None" means no superconvergence results. The other sets are defined below.

$\text{SPH} = \{\ (\xi,\eta,\zeta) \in K \mid \xi,\eta,\zeta = -1, 0, 1\ \}$;

$\text{HF}_n = \{\ (\xi,\eta,\zeta) \in K \mid \phi_{n+1}(\xi) = 0,\ \phi_{n+1}(\eta) = 0,\ \phi_{n+1}(\zeta) = 0\ \}$;

$\text{HD}_n = \{\ (\xi,\eta,\zeta) \in K \mid P_n(\xi) = 0\ \}$;

$\text{HSet1} = \{\ (\pm\frac{\sqrt{-9+6\sqrt{3}}}{3}, \pm\frac{\sqrt{-9+6\sqrt{3}}}{3}, \pm\frac{\sqrt{-9+6\sqrt{3}}}{3}),$
$(\pm\frac{\sqrt{-9+6\sqrt{3}}}{3}, \pm\frac{\sqrt{-9+6\sqrt{3}}}{3}, \pm\frac{\sqrt{-225+210\sqrt{3}}}{15}),$
$(\pm\frac{\sqrt{-9+6\sqrt{3}}}{3}, \pm\frac{\sqrt{-225+210\sqrt{3}}}{15}, \pm\frac{\sqrt{-9+6\sqrt{3}}}{3}),$
$(\pm\frac{\sqrt{-225+210\sqrt{3}}}{15}, \pm\frac{\sqrt{-9+6\sqrt{3}}}{3}, \pm\frac{\sqrt{-9+6\sqrt{3}}}{3})\ \}$;

$\text{HSet2} = \{\ (\xi,\eta,\zeta) \in K \mid \xi = 0\ \} \cup \{\ (\pm\sqrt{3/5}, \pm\sqrt{1/3}, \pm\sqrt{1/3})\ \}$;

$\text{HSet3} = \{\ (0,0,0), (0,\pm 1, 0), (0,0,\pm 1), (0,\pm 1,\pm 1)\ \}$;

$\text{HSet4} = \{\ (\xi,\eta,\zeta) \in K \mid \xi = 0, \quad \text{or } \xi = \pm\sqrt{\frac{1+6\eta^2}{5}} \text{ and } \zeta = \pm\eta\ \}$;

where P_n is the Legendre polynomial of degree n, and ϕ_n is defined in (2.1). Superconvergence results of the η- and ζ-derivatives can be obtained analogously.

6.2. Pentahedral elements. Tables 3 and 4 are devoted to superconvergence results for Lagrangian and Serendipity pentahedral elements. Elements of degree up to 5 are studied.

TABLE 3
Function value superconvergence results of pentahedral FE.

	Lagrangian pentahedral FE		Serendipity pentahedral FE	
n	Poisson	Laplace	Poisson	Laplace
2	SPH	PF_2	SPH	PF_2
3	None	PF_3	None	None
4	PSet1	PF_4	SPH	SPH
5	None	PF_5	None	None

TABLE 4
Derivative superconvergence results of pentahedral FE.

	Lagrangian pentahedral FE		Serendipity pentahedral FE	
n	Poisson	Laplace	Poisson	Laplace
1	ξ- : PSet2 ζ- : $PD1_1$	ξ- : $PD2_1$ ζ- : $PD1_1$	ξ- : PSet2 ζ- : $PD1_1$	ξ- : PSet2 ζ- : $PD1_1$
2	ξ- : PSet3 ζ- : $PD1_2$	ξ- : $PD2_2$ ζ- : $PD1_2$	ξ- : PSet3 ζ- : $PD1_2$	ξ- : $PD2_2$ ζ- : $PD1_2$
3	ξ- : PSet2 ζ- : $PD1_3$	ξ- : $PD2_3$ ζ- : $PD1_3$	ξ- : PSet2 ζ- : $\{\zeta = 0\}$	ξ- : PSet2 ζ- : PSet6
4	ξ- : None ζ- : $PD1_4$	ξ- : $PD2_4$ ζ- : $PD1_4$	ξ- : None ζ- : None	ξ- : None ζ- : None
5	ξ- : PSet2 ζ- : $PD1_5$	ξ- : $PD2_5$ ζ- : $PD1_5$	ξ- : PSet4 ζ- : PSet5	ξ- : PSet4 ζ- : PSet5

The sets in the tables are defined as

$\mathrm{PF}_n = \{(\xi,\eta,\zeta) \mid (\xi,\eta) \in \mathrm{FSR}_n, \quad \phi_{n+1}(\zeta) = 0\}$;
$\mathrm{PD1}_n = \{(\xi,\eta,\zeta) \mid P_n(\zeta) = 0\}$;
$\mathrm{PD2}_n = \{(\xi,\eta,\zeta) \mid (\xi,\eta) \in \mathrm{DSR}_{\xi,n}\}$;
$\mathrm{PSet1} = \{(\xi,\eta,\zeta) \mid \xi,\eta = \pm 1, 0, \quad \phi_5(\zeta) = 0\}$;
$\mathrm{PSet2} = \{(\xi,\eta,\zeta) \mid \xi = 0,\ \eta = \pm 1\}$;
$\mathrm{PSet3} = \{(\xi,\eta,\zeta) \mid \xi = \pm\sqrt{3}/3,\ \eta = \pm 1\}$;
$\mathrm{PSet4} = \{(0,\pm 1,\pm 1),\ (0,\pm 1, 0)\}$;
$\mathrm{PSet5} = \{(0,0,0),\ (\pm 1,0,0),\ (0,\pm 1,0),\ (\pm 1,\pm 1,0)\}$;
$\mathrm{PSet6} = \{(\xi,\eta,\zeta) \mid \zeta = 0\} \cup \{\pm(1-\frac{\sqrt{3}}{3}, -1+\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{90-50\sqrt{3}}}{5})\}$;

where FSR_n is the collection of the nth order 2D regular pattern function value superconvergence points of the Laplace equation, and $\mathrm{DSR}_{\xi,n}$ is the set of nth order 2D regular pattern ξ-derivative superconvergence points of the Laplace equation [18]. Superconvergence results of the η-derivatives are analogous to those of the ξ-derivatives.

6.3. Tetrahedral elements. Superconvergence results for tetrahedral elements of degree up to 4 are collected in Tables 5 and 6.

Here, TSet1 is the set of vertices and midpoints of edges of K; TSet2 is the set of vertices of K and midpoints of diagonal edges of K (e.g., edges $l_{1,21}$, $l_{1,23}$, $l_{1,25}$ in Figure 4); TSet3 is the set of midpoints of the edges of K which are parallel to the ξ-axis; and TSet4 is the set of second order Gaussian points along the edges of

TABLE 5
Function value superconvergence results of tetrahedral FE.

	Scheme 1	Scheme 2	
n	Poisson & Laplace	Poisson	Laplace
2	TSet1	TSet2	TSet1*
3	None	None	None
4	TSet1	TSet2	TSet1*

TABLE 6
ξ-derivative superconvergence results of tetrahedral FE.

	Scheme 1	Scheme 2	
n	Poisson & Laplace	Poisson	Laplace
1	TSet3	None	TSet3*
2	TSet4	None	TSet4*
3	TSet3	None	TSet3*
4	None	None	None

K which are parallel to the ξ-axis. Note that TSet$m$ and TSetm^* are different due to the different partition schemes. For η- and ζ-derivatives, superconvergence results can be obtained analogously.

In addition, under partition scheme 2, if we consider derivatives along diagonal directions (e.g., directions of edges $l_{1,21}$, $l_{1,23}$, $l_{1,25}$ in Figure 4), there are superconvergence results. In particular, when $n = 1$ and 3, the midpoints of the edges parallel to the tangential direction are superconvergence points; when $n = 2$, the second order Gaussian points of the tangential direction edges are superconvergence points; when $n = 4$, there are no superconvergence points. The results hold for both the Poisson and the Laplace equations.

7. Concluding remarks. In this article, a systematic and analytic approach in determining 3D FE superconvergence points is presented. We investigate hexahedral, pentahedral, and tetrahedral elements, which are widely used in engineering applications. The results for the Poisson equation can be applied to many problems which have second order elliptic operators, including some nonlinear problems. The approach can be used to locate superconvergence points of any degree and provides conclusive results. We conclude this article by the following remarks.

Remark 7.1. In most 3D elements, superconvergence points of both the Poisson and Laplace equations are the same, as demonstrated in section 6. This is in contrast with the 2D cases, where there are many more superconvergence points of the Laplace equation than that of the Poisson equation for higher order elements. The reason is that, when $n \geq 2$, there are only two basis functions in the space of harmonic polynomials for each degree n in two dimensions, while there are $2n+3$ basis functions in the space of harmonic polynomials for each degree n in three dimensions. The likelihood of $2n + 3$ polynomials passing a common point is much smaller than the probability of two polynomials intersecting.

Remark 7.2. Although some superconvergence points can also be predicted by the symmetry theory and the tensor-product technique, we nevertheless provide conclusive results and demonstrate that in some cases symmetry points and/or points obtained by tensor-product are the only superconvergence points, while in other cases there are more superconvergence points which are not these types of points.

Remark 7.3. As indicated in [27, Chap. 14], many recovery techniques utilize a priori knowledge of natural superconvergence points. Therefore, the results in this paper are not only interesting by themselves but also good "hints" for more flexible and robust recovery methods.

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