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CAN WE HAVE SUPERCONVERGENT GRADIENT RECOVERY UNDER ADAPTIVE MESHES?*

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Abstract. We study adaptive finite element methods for elliptic problems with domain corner singularities. Our model problem is the two-dimensional Poisson equation. Results of this paper are twofold. First, we prove that there exists an adaptive mesh (gauged by a discrete mesh density function) under which the recovered gradient by the polynomial preserving recovery (PPR) is superconvergent. Second, we demonstrate by numerical examples that an adaptive procedure with an a posteriori error estimator based on PPR does produce adaptive meshes that asatisfy our mesh density assumption, and the recovered gradient by PPR is indeed superconvergent in the adaptive process.

Key words. finite element method, adaptive, superconvergence, gradient recovery

AMS subject classifications. 65N30, 65N15, 45K20

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1. Introduction.

Let $\Omega \subset R^2$ be a bounded polygon with boundary $\partial\Omega$. Consider the following Dirichlet boundary problem: Find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$A(u, v) = \int_\Omega \nabla u \cdot \nabla v = f(v) \quad \forall v \in H_0^1(\Omega), \tag{1.1}$$

where $f \in H^{-1}(\Omega)$.

It is well known that the solution u may have singularities at corners of Ω. Since the treatment of multiple singular points is no different from a simple one, without loss of generality we assume that the solution u has a singularity at the origin O and can be decomposed as a sum of a singular part and a smooth part:

$$u = v + w, \tag{1.2}$$

where

$$\left|\frac{\partial^m v}{\partial x^i \partial y^{m-i}}\right| \lesssim r^{\delta - m} \text{ and } \left|\frac{\partial^m w}{\partial x^i \partial y^{m-i}}\right| \lesssim 1, \quad m = 1, \dots, k+2,\ i = 0, \dots, m, \tag{1.3}$$

where $r = \sqrt{x^2 + y^2}$ and $0 < \delta < k + 1$ is a constant. Here $k = 1$ for linear finite element methods and $k = 2$ for quadratic finite element methods.

Next, we briefly explain the rationale of the above regularity assumption. When Ω is a polygonal domain, the solution of the Poisson equation with the Dirichlet

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boundary condition,

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = g,$$

with sufficiently smooth data f and g, has the following decomposition (see, e.g., [3] and [11]), at a corner with angle ω:

$$u(r,\theta) = \sum_{j=1}^{J} c_j r^{\alpha_j} \ln^{s_j} r \sin \alpha_j \theta + w, \quad \alpha_j = \frac{j\pi}{\omega},$$

where w is smoother than the terms in the sum, and

$$s_j = \begin{cases} 1 & \alpha_j \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Especially for the L-shaped domain, $\omega = 3\pi/2$ at the re-entrance corner, and the expansion is

$$u = c_1 r^{2/3} \sin \frac{2}{3}\theta + c_2 r^{4/3} \sin \frac{4}{3}\theta + c_3 r^2 \ln r \sin 2\theta + c_4 r^{8/3} \sin \frac{8}{3}\theta + w,$$

with $w \in W^3_\infty(\Omega)$. For a cracked domain, $\omega = 2\pi$ at the crack tip and the expansion is

$$u = c_1 r^{1/2} \sin \frac{1}{2}\theta + c_2 2r \ln r \sin \theta + c_3 r^{3/2} \sin \frac{3}{2}\theta$$
$$+ c_4 r^2 \ln r \sin 2\theta + c_5 r^{5/2} \sin \frac{5}{2}\theta + c_6 r^3 \ln r \sin 3\theta + w,$$

with $w \in W^3_\infty(\Omega)$. More terms are needed in the expansion if we want higher regularity on w. These are the two cases we shall test numerically in the last section.

Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω, $\mathcal{E}_h$ be the set of all interior edges, and $\mathcal{N}_h$ be the set of all nodal points. Assume that the origin $O \in \mathcal{N}_h$. Remember that any triangle $\tau \in \mathcal{M}_h$ is considered to be closed. Let

$$V_h^k = \left\{ v_h : \ v_h \in H^1(\Omega), v_h|_\tau \in P_k(\tau) \right\}, \quad k = 1, 2,$$

be the conforming finite element space associated with $\mathcal{M}_h$, and let $\overset{\circ}{V}{}_h^k = V_h^k \bigcap H_0^1(\Omega)$. Here P_k denotes the set of polynomials with degree $\leq k$. Denote by $I_h^k : C(\bar{\Omega}) \to V_h^k$ the standard finite element interpolation operator. The finite element solution $u_h \in V_h^k$ satisfies $u_h = I_h^k u$ on $\partial\Omega$ and

$$A(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h = f(v_h) \quad \forall v_h \in \overset{\circ}{V}{}_h^k. \tag{1.4}$$

In adaptive finite element methods, the convergence rate is measured by the total number of degrees of freedom N, since the mesh is not quasi-uniform. For a two-dimensional second-order elliptic equation, the optimal convergence rates are

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \lesssim \begin{cases} N^{-1/2}, & k = 1, \\ N^{-1}, & k = 2, \end{cases} \tag{1.5}$$

where $k = 1$ for the linear element and $k = 2$ for the quadratic.

The theoretical development of residual-type error estimates is now in its maturity. For the early literature, readers are referred to [1, 4, 10, 21] and references therein. Starting from the fundamental work of [9], in the last decade the convergence proof of residual-based adaptive finite element method has been well established; see, e.g., [2, 8, 17, 19]. On the contrary, there is no convergence proof for using recovery-based error estimators. Nevertheless, by shifting the error estimator from residual based to recovery based, we have obtained the same numerical convergence rate following the same mark-up and refinement procedure for two model problems—the Poisson equation on the L-shaped domain and cracked square. Theoretically, we are able to prove that there exists an adaptive mesh satisfying a discrete mesh density condition such that the convergence rate (1.5) can be established. Moreover, under the same mesh density condition, the recovered gradient $G_h u_h$ is superconvergent in the sense that

$$\|\nabla u - G_h u_h\|_{L^2(\Omega)} \lesssim \begin{cases} N^{-1/2-\rho}, & k = 1, \\ N^{-1-\rho}, & k = 2, \end{cases} \tag{1.6}$$

where $\rho > 0$ is a constant, which depends on the quality of the adaptive mesh, and $G_h : V_h^k \to V_h^k \times V_h^k$ is the recovery operator. Now the question is: Is the condition required by our theory practical? We demonstrate that the meshes generated by the standard adaptive procedure in both of our model problems indeed satisfies the mesh density condition.

In recent years there have been some superconvergence results for a recovered gradient [5, 15, 8, 16, 20, 23, 24, 25, 26, 28]. All of them assumed at least $u \in H^3(\Omega) \cap W^2_\infty(\Omega)$ (a condition that rules out domains with a re-entrant corner) and required some stronger (than we required here) mesh conditions. Our current work fills in this gap. To the best of our knowledge, this is the first theoretical superconvergence proof for real-life adaptive meshes.

Some further theoretical results about recovery techniques and recovery-type error estimators can be found in [1, 7, 22, 13].

Based on the estimate (1.6), we suggest that, even for residual-type adaptive method, a gradient recovery procedure at the very last mesh would dramatically improve the numerical gradient.

Throughout the paper, we use the notation $A \lesssim B$ to represent the inequality $A \leq$ *constant* $\times B$, where the *constant* may depend only on the minimum angle of the triangles in the mesh $\mathcal{M}_h$, the constant δ, and the domain Ω. The notation $A \eqsim B$ is equivalent to the statement $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries. Following the discussion in [8], we consider in Figure 2.1 an edge e, two elements τ and τ' sharing e, and $\Omega_e = \tau \cup \tau'$ the patch of e. For an element $\tau \subset \Omega_e$, θ_e denotes the angle opposite of the edge e, h_e, h_{e+1}, and h_{e-1} denote the lengths of the three edges of τ. The subscript $e+1$ or $e-1$ is for orientation. All triangles in the triangulation are orientated counterclockwise. $\boldsymbol{t}_e$ is the unit tangent vector of e with counterclockwise orientation and $\boldsymbol{n}_e$ is the unit outward normal vector. An index $'$ is added for the corresponding quantities in τ'. Notice that $\boldsymbol{t}_e = -\boldsymbol{t}'_e$ and $\boldsymbol{n}_e = -\boldsymbol{n}'_e$ because of the orientation. For any $\tau \in \mathcal{M}_h$, we denote by h_τ its diameter and by r_τ the distance from the origin to the barycenter of τ, and by $|\tau|$ the area of the triangle τ. For any $e \in \mathcal{E}_h$, let r_e be the distance from the origin O to the midpoint of e.

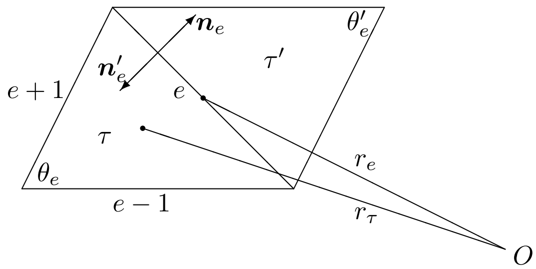

FIG. 2.1. *Notation in the patch* Ω_e.

Let $e \in \mathcal{E}_h$ be an interior edge. Recall that Ω_e, the patch of e, consists of two adjacent triangles sharing e. We say that Ω_e is an ε approximate parallelogram if the lengths of any two opposite edges differ by at most ε.

Definition. *The triangulation* $\mathcal{M}_h$ *is said to satisfy* Condition (α, σ, μ) *if there exist constants* $\alpha > 0$*,* $\sigma \geq 0$*, and* $\mu > 0$ *such that the interior edges can be separated into two parts* $\mathcal{E}_h = \mathcal{E}_{1,h} \oplus \mathcal{E}_{2,h}$*:* Ω_e *forms an* $O\big(h_e^{1+\alpha}/r_e^{\alpha+\mu(1-\alpha)}\big)$ *parallelogram for* $e \in \mathcal{E}_{1,h}$ *and the number of edges in* $\mathcal{E}_{2,h}$ *satisfies* $\#\mathcal{E}_{2,h} \lesssim N^\sigma$*.*

Remark 2.1. The meaning of *Condition* (α, σ, μ) is the following. The edges can be grouped into "good" ($\mathcal{E}_{1,h}$) and "bad" ($\mathcal{E}_{2,h}$), where the number of bad edges is much smaller than that of good edges. The ratio is

$$\frac{\#\mathcal{E}_{2,h}}{\#\mathcal{E}_{1,h}} \lesssim \frac{N^\sigma}{N} = \frac{1}{N^{1-\sigma}}.$$

When $r_e = O(1)$, i.e., an edge e is far away from the singular point O, more restrictions are put on the adjacent triangles with the common edge e. This condition requires that they form an $O(h_e^{1+\alpha})$ parallelogram, which is the same as in previous works [20, 23, 25, 26]. When e is in a neighborhood of O, where $r_e^{1+\mu(1-\alpha)/\alpha} \lesssim h_e$, the condition $O(h_e)$ implies $O\big(h_e^{1+\alpha}/r_e^{\alpha+\mu(1-\alpha)}\big)$. In other words, two adjacent triangles that share e are allow to distort $O(h_e)$ from a parallelogram, which implies no restriction on them. Roughly speaking, the number of edges in $\mathcal{E}_{1,h}$ that have no restriction imposed is $O(N^{1-\alpha})$ if $h_\tau \eqsim r_\tau^{1-\mu}\underline{h}^\mu$ for any $\tau \in \mathcal{M}_h$. Here $\underline{h}$ and μ are positive constants. An explanation is given below after Lemma 2.1.

We see from the above discussion that the closer we are to the singular point, the less restriction is imposed on the mesh. Indeed, for an adaptively refined mesh, the closer we are to the singular point, the worse the mesh quality is in terms of forming parallelogram triangular pairs.

LEMMA 2.1. *Assume that* $h_\tau \eqsim r_\tau^{1-\mu}\underline{h}^\mu$ *for any* $\tau \in \mathcal{M}_h$*, where* $\underline{h}$ *and* μ *are positive constants. Then the total number of degrees of freedom* N *of the finite element equation* (1.4) *satisfies*

$$N \eqsim \frac{1}{\underline{h}^{2\mu}}. \tag{2.1}$$

Proof.

$$\begin{aligned} N &\eqsim \sum_{\tau\in\mathcal{M}_h} \frac{h_\tau^2}{h_\tau^2} \eqsim \frac{1}{\underline{h}^{2\mu}} \sum_{\tau\in\mathcal{M}_h} \frac{1}{r_\tau^{2-2\mu}} \cdot |\tau| \\ &\eqsim \frac{1}{\underline{h}^{2\mu}} \int_\Omega \frac{1}{r^{2-2\mu}} \eqsim \frac{1}{\underline{h}^{2\mu}} \int_0^1 \frac{1}{r^{2-2\mu}} \cdot r\,\mathrm{d}r \eqsim \frac{1}{\underline{h}^{2\mu}}. \end{aligned}$$

This completes the proof of the lemma. □

Remark 2.2. For the linear element, $\mu = \delta/2$, $N \eqsim 1/\underline{h}^{\delta}$, and for the quadratic element $\mu = \delta/3$, $N \eqsim 1/\underline{h}^{2\delta/3}$. The condition $h_\tau \eqsim r_\tau^{1-\mu}\underline{h}^{\mu}$ can be viewed as a discrete mesh density function. The positive number $\underline{h} \eqsim \min_{\tau\in\mathcal{M}_h} h_\tau$, is the size of the minimum element because for an element τ neighboring O, $r_\tau \eqsim h_\tau$ and the condition $h_\tau \eqsim r_\tau^{1-\mu}\underline{h}^{\mu}$ implies that $h_\tau \eqsim \underline{h}$. It is clear that the condition $h_\tau \eqsim r_\tau^{1-\mu}\underline{h}^{\mu}$ for any $\tau \in \mathcal{M}_h$ is equivalent to the condition $h_e \eqsim r_e^{1-\mu}\underline{h}^{\mu}$ for any $e \in \mathcal{E}_h$. We recall that *Condition* (α, σ, μ) means no restriction on Ω_e if $r_e^{1+\mu(1-\alpha)/\alpha} \lesssim h_e$. Furthermore, if $h_\tau \eqsim r_\tau^{1-\mu}\underline{h}^{\mu}$, i.e., $h_e \eqsim r_e^{1-\mu}\underline{h}^{\mu}$, then $r_e \lesssim \underline{h}^{\alpha}$. Therefore if the mesh $\mathcal{M}_h$ satisfies *Condition* (α, σ, μ) and $h_\tau \eqsim r_\tau^{1-\mu}\underline{h}^{\mu}$, then no restriction is imposed on edges within the ball of radius $R \lesssim \underline{h}^{\alpha}$. The number of edges in the ball is $O(N^{1-\alpha})$ by an argument similar to the proof of Lemma 2.1.

3. Superconvergence between the finite element solution and linear interpolant. We now define a quadratic interpolant of ϕ based on moment conditions on edges. Let $\phi_Q = \Pi_Q\phi$ be a quadratic element defined by

$$(3.1) \qquad (\Pi_Q\phi)(z) = \phi(z), \quad \text{and} \quad \int_e \Pi_Q\phi = \int_e \phi \quad \forall z \in \mathcal{N}_h, e \in \mathcal{E}_h.$$

The following fundamental identity is proved in [8] for $v_h \in P_1(\tau)$:

$$(3.2) \qquad \int_\tau \nabla(\phi - \phi_I)\cdot\nabla v_h = \sum_{e\subset\partial\tau}\left(\beta_e\int_e \frac{\partial^2\phi_Q}{\partial \boldsymbol{t}_e^2}\frac{\partial v_h}{\partial \boldsymbol{t}_e} + \gamma_e\int_e \frac{\partial^2\phi_Q}{\partial \boldsymbol{t}_e\partial \boldsymbol{n}_e}\frac{\partial v_h}{\partial \boldsymbol{t}_e}\right),$$

where

$$(3.3) \qquad \beta_e = \frac{1}{12}\cot\theta_e(h_{e+1}^2 - h_{e-1}^2), \quad \gamma_e = \frac{1}{3}\cot\theta_e\,|\tau|,$$

and $\phi_I \in P_1(\tau)$ is the linear interpolant of ϕ on τ. The following lemma is a simple modification of [8, Lemma 2.13].

LEMMA 3.1. *Let $\boldsymbol{m}_e$ denote $\boldsymbol{t}_e$ or $\boldsymbol{n}_e$. Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha, \sigma, \delta/2)$ *with $0 < \alpha \le 1$ and $0 \le \sigma < 1$. For any interior edge $e \in \mathcal{M}_h$ and two elements τ, $\tau' \subset \Omega_e$, we have*

$$(3.4) \quad |\beta_e| + |\beta_e'| \lesssim h_e^2, \quad |\gamma_e| + |\gamma_e'| \lesssim h_e^2 \quad \forall e \in \mathcal{E}_h;$$

$$(3.5) \quad |\beta_e - \beta_e'| \lesssim h_e^{2+\alpha}/r_e^{\alpha+\delta(1-\alpha)/2}, \quad |\gamma_e - \gamma_e'| \lesssim h_e^{2+\alpha}/r_e^{\alpha+\delta(1-\alpha)/2} \quad \forall e \in \mathcal{E}_{1,h};$$

$$(3.6) \quad \int_e \frac{\partial^2\phi}{\partial \boldsymbol{t}_e\partial \boldsymbol{m}_e}\frac{\partial v_h}{\partial \boldsymbol{t}_e} \lesssim |\phi|_{W^{2,\infty}(e)}\,\|\nabla v_h\|_{L^2(\tau)};$$

$$(3.7) \quad \int_e \frac{\partial^2(\phi - \phi_Q)}{\partial \boldsymbol{t}_e\partial \boldsymbol{m}_e}\frac{\partial v_h}{\partial \boldsymbol{t}_e} \lesssim |\phi|_{H^3(\tau)}\,\|\nabla v_h\|_{L^2(\tau)}.$$

Proof. The arguments for (3.4), (3.5), and (3.6) are trivial, and that for (3.7) follows from the trace theorem and the standard error estimate $|\phi - \phi_Q|_{H^2(\tau)} \lesssim h_\tau\,|\phi|_{H^3(\tau)}$. □

To deal with the singularity at the origin O we introduce the following lemma. Recall that v is the singular part of the decomposition $u = v + w$.

Lemma 3.2. *Let $\mathcal{M}^O = \{\tau \in \mathcal{M}_h : \text{ the origin } O \in \partial\tau\}$ be the set of elements with one vertex at O. Then,*

$$\|\nabla v - \nabla v_I\|_{L^2(\tau)} \lesssim h_\tau^\delta \quad \forall \tau \in \mathcal{M}^O,$$

where $v_I = I_h^1 v$ is the linear interpolant of v.

Proof.

$$\|\nabla v - \nabla v_I\|_{L^2(\tau)} \lesssim \|\nabla v\|_{L^2(\tau)} + \|\nabla v_I\|_{L^2(\tau)}. \tag{3.8}$$

It follows from (1.3) that

$$\|\nabla v\|_{L^2(\tau)} = \left(\int_\tau |\nabla v|^2\right)^{1/2} \lesssim \left(\int_\tau r^{2\delta-2}\right)^{1/2} \lesssim \left(\int_0^{h_\tau} r^{2\delta-2} r\,\mathrm{d}r\right)^{1/2} \lesssim h_\tau^\delta. \tag{3.9}$$

Since $\nabla C = 0$, for any constant C, we have,

$$\begin{aligned}\|\nabla v_I\|_{L^2(\tau)} &= \|\nabla(v_I - v(O))\|_{L^2(\tau)} \lesssim h_\tau \max_{z\in\mathcal{N}_h\cap\tau} \left|\nabla\big(v_I - v(O)\big)(z)\right| \\ &\lesssim h_\tau \frac{1}{h_\tau} \max_{z\in\mathcal{N}_h\cap\tau} |v(z) - v(O)| \\ &= \max_{z\in\mathcal{N}_h\cap\tau} \left|\int_0^1 \frac{d}{dt} v(zt)\mathrm{d}t\right| = \max_{z\in\mathcal{N}_h\cap\tau} \left|\int_0^1 z\cdot\nabla v(zt)\mathrm{d}t\right|.\end{aligned}$$

Noting that $|z| \lesssim h_\tau$ for $\tau \in \mathcal{M}^O$, it follows from assumption (1.3) that

$$\|\nabla v_I\|_{L^2(\tau)} \lesssim \int_0^1 h_\tau \cdot (h_\tau t)^{\delta-1}\mathrm{d}t \lesssim h_\tau^\delta. \tag{3.10}$$

The proof is completed by combining (3.8)–(3.10). ∎

Lemma 3.3. *Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha, \sigma, \delta/2)$ *with $0 < \alpha \le 1$ and $0 \le \sigma < 1$, and that $h_\tau \eqsim r_\tau^{1-\delta/2}\underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then for any $v_h \in \overset{\circ}{V}{}_h^1$,*

$$\left|\int_\Omega \nabla(u - u_I)\cdot\nabla v_h\right| \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right), \tag{3.11}$$

where $u_I = I_h^1 u \in V_h^1$ is the piecewise linear interpolant of u.

Proof. From the decomposition $u = v + w$,

$$\int_\Omega \nabla(u - u_I)\cdot\nabla v_h = \int_\Omega \nabla(v - v_I)\cdot\nabla v_h + \int_\Omega \nabla(w - w_I)\cdot\nabla v_h, \tag{3.12}$$

where $v_I = I_h^1 v$ and $w_I = I_h^1 w$ are the linear interpolants of v and w, respectively.

We first estimate $\int_\Omega \nabla(v - v_I)\cdot\nabla v_h$. Let $\mathcal{E}^O = \{e \in \mathcal{E}_h : e \subset \partial\tau \text{ the origin } O \in \tau\}$ and $\partial\mathcal{E}^O = \{e \in \mathcal{E}^O : O \notin e\}$. Recall that $\mathcal{M}^O$ is the set of elements with one vertex at O. Applying (3.2),

$$\begin{aligned}\int_\Omega \nabla(v - v_I)\cdot\nabla v_h &= \sum_{\tau\in\mathcal{M}_h} \int_\tau \nabla(v - v_I)\cdot\nabla v_h = \sum_{\tau\in\mathcal{M}^O} \int_\tau \nabla(v - v_I)\cdot\nabla v_h \\ &\quad + \sum_{\tau\in\mathcal{M}_h\setminus\mathcal{M}^O} \sum_{e\subset\partial\tau} \left(\beta_e \int_e \frac{\partial^2 v_Q}{\partial \boldsymbol{t}_e^2}\frac{\partial v_h}{\partial \boldsymbol{t}_e} + \gamma_e \int_e \frac{\partial^2 v_Q}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e}\frac{\partial v_h}{\partial \boldsymbol{t}_e}\right) \\ &= I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{3.13}$$

where

$$I_j = \sum_{e\in\mathcal{E}_{j,h}\setminus\mathcal{E}^O} \left[(\beta_e - \beta_e') \int_e \frac{\partial^2 v}{\partial \boldsymbol{t}_e^2} \frac{\partial v_h}{\partial \boldsymbol{t}_e} + (\gamma_e - \gamma_e') \int_e \frac{\partial^2 v}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e} \frac{\partial v_h}{\partial \boldsymbol{t}_e} \right.$$
$$+ \beta_e \int_e \frac{\partial^2 (v_Q - v)}{\partial \boldsymbol{t}_e^2} \frac{\partial v_h}{\partial \boldsymbol{t}_e} + \gamma_e \int_e \frac{\partial^2 (v_Q - v)}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e} \frac{\partial v_h}{\partial \boldsymbol{t}_e}$$
$$\left. + \beta_e' \int_e \frac{\partial^2 (v - v_Q)}{\partial \boldsymbol{t}_e^2} \frac{\partial v_h}{\partial \boldsymbol{t}_e} + \gamma_e' \int_e \frac{\partial^2 (v - v_Q)}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e} \frac{\partial v_h}{\partial \boldsymbol{t}_e} \right], \quad j = 1, 2,$$

$$I_3 = \sum_{\tau\in\mathcal{M}^O} \int_\tau \nabla (v - v_I) \cdot \nabla v_h,$$

$$I_4 = \sum_{e\in\partial\mathcal{E}^O} \left(\beta_e \int_e \frac{\partial^2 v_Q}{\partial \boldsymbol{t}_e^2} \frac{\partial v_h}{\partial \boldsymbol{t}_e} + \gamma_e \int_e \frac{\partial^2 v_Q}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e} \frac{\partial v_h}{\partial \boldsymbol{t}_e} \right).$$

First, I_3 can be estimated by Lemma 3.2 and the fact that $h_\tau \eqsim \underline{h}$ for $\tau \in \mathcal{M}^O$:

$$|I_3| \lesssim \underline{h}^\delta \sum_{\tau\in\mathcal{M}^O} \|\nabla v_h\|_{L^2(\tau)} \lesssim \underline{h}^\delta \|\nabla v_h\|_{L^2(\Omega)}. \tag{3.14}$$

Second, I_4 can be estimated by Lemma 3.1, assumption (1.3), and the fact that $h_e \eqsim r_e \eqsim \underline{h}$ for $e \in \partial\mathcal{E}^O$:

$$\begin{aligned} |I_4| &\lesssim \sum_{e\in\partial\mathcal{E}^O} h_e^2 \left(|v|_{W^{2,\infty}(e)} + |v|_{H^3(\tau:\,\tau\in\Omega_e,\tau\notin\mathcal{M}^O)} \right) \|\nabla v_h\|_{L^2(\tau:\,\tau\in\Omega_e,\tau\notin\mathcal{M}^O)} \\ &\lesssim \sum_{e\in\partial\mathcal{E}^O} h_e^2 \big(r_e^{\delta-2} + h_e r_e^{\delta-3} \big) \|\nabla v_h\|_{L^2(\tau:\,\tau\in\Omega_e,\tau\notin\mathcal{M}^O)} \\ &\lesssim \underline{h}^\delta \sum_{e\in\partial\mathcal{E}^O} \|\nabla v_h\|_{L^2(\tau:\,\tau\in\Omega_e,\tau\notin\mathcal{M}^O)} \lesssim \underline{h}^\delta \|\nabla v_h\|_{L^2(\Omega)}. \end{aligned} \tag{3.15}$$

Next we estimate I_1. Notice that $h_e \eqsim h_\tau$ and $r_e \eqsim r_\tau$ for $\tau \subset \Omega_e$ and $e \in \mathcal{E}_{1,h}\setminus\mathcal{E}^O$. It follows from Lemma 3.1 and assumption (1.3) that

$$\begin{aligned} |I_1| &\lesssim \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O} \left[\frac{h_e^{2+\alpha}}{r_e^{\alpha+\delta(1-\alpha)/2}} r_e^{\delta-2} + h_e^2 h_\tau r_\tau^{\delta-3} \right] \|\nabla v_h\|_{L^2(\tau:\,\tau\in\Omega_e)} \\ &\lesssim \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O} \left[h_e^{2+\alpha} r_e^{\delta-2-\alpha-\delta(1-\alpha)/2} + h_e^3 r_e^{\delta-3} \right] \|\nabla v_h\|_{L^2(\tau:\,\tau\in\Omega_e)} \\ &\lesssim \left\{ \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O} \left[h_e^2 h_e^{2+2\alpha} r_e^{2\delta-4-2\alpha-\delta(1-\alpha)} + h_e^2 h_e^4 r_e^{2\delta-6} \right] \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \\ &\lesssim \left\{ \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O} \left[h_e^2 \underline{h}^{\delta(1+\alpha)} r_e^{(2-\delta)(1+\alpha)} r_e^{2\delta-4-2\alpha-\delta(1-\alpha)} + h_e^2 \underline{h}^{2\delta} r_e^{4-2\delta} r_e^{2\delta-6} \right] \right\}^{1/2} \\ &\quad \times \|\nabla v_h\|_{L^2(\Omega)}. \end{aligned}$$

Here we have used $h_e \eqsim r_e^{1-\delta/2}\underline{h}^{\delta/2}$ to derive the last inequality. Therefore

$$
\begin{aligned}
I_1 &\lesssim \left\{\underline{h}^{\delta(1+\alpha)} \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O} h_e^2 r_e^{-2}\right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \lesssim \left\{\underline{h}^{\delta(1+\alpha)} \sum_{\tau\in\mathcal{M}_h\setminus\mathcal{M}^O} h_\tau^2 r_\tau^{-2}\right\}^{1/2} \\
&\quad \times \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \left\{\underline{h}^{\delta(1+\alpha)} \sum_{\tau\in\mathcal{M}_h\setminus\mathcal{M}^O} \int_\tau r^{-2}\right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \lesssim \left\{\underline{h}^{\delta(1+\alpha)} \int_{\underline{h}}^1 r^{-1}\,\mathrm{d}r\right\}^{1/2} \\
&\quad \times \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \underline{h}^{\delta(1+\alpha)/2}(|\ln\underline{h}|^{1/2}) \|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned} \tag{3.16}
$$

Finally, we estimate I_2. Notice that $h_e \lesssim r_e$ for $e \notin \mathcal{E}^O$. It follows from Lemma 3.1 and assumption (1.3) that

$$
\begin{aligned}
|I_2| &\lesssim \sum_{e\in\mathcal{E}_{2,h}\setminus\mathcal{E}^O} \left[h_e^2 r_e^{\delta-2} + h_e^2 h_\tau r_\tau^{\delta-3}\right] \|v_h\|_{L^2(\tau:\,\tau\in\Omega_e)} \\
&\lesssim \sum_{e\in\mathcal{E}_{2,h}\setminus\mathcal{E}^O} h_e^2 r_e^{\delta-2} \|v_h\|_{L^2(\tau:\,\tau\in\Omega_e)} \\
&\lesssim \left\{\sum_{e\in\mathcal{E}_{2,h}\setminus\mathcal{E}^O} h_e^4 r_e^{2\delta-4}\right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \underline{h}^\delta \left\{\sum_{e\in\mathcal{E}_{2,h}\setminus\mathcal{E}^O} 1\right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned}
$$

Here we have used $h_e \eqsim r_e^{1-\delta/2}\underline{h}^{\delta/2}$ to derive the last inequality. Therefore

$$
|I_2| \lesssim \underline{h}^\delta \left\{\#\mathcal{E}_{2,h}\right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \lesssim \underline{h}^\delta \left\{N^\sigma\right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)}. \tag{3.17}
$$

From Lemma 2.1, $\underline{h}^\delta \eqsim 1/N$, $|\ln\underline{h}| \eqsim \ln N$. Combining (3.13)–(3.17) we have

$$
\begin{aligned}
\left|\int_\Omega \nabla(v - v_I)\cdot\nabla v_h\right| &\lesssim \left(\underline{h}^{\delta(1+\alpha)/2}(|\ln\underline{h}|^{1/2}) + \underline{h}^\delta\left\{N^\sigma\right\}^{1/2}\right) \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right).
\end{aligned} \tag{3.18}
$$

Now we turn to the estimate for $\int_\Omega \nabla(w - w_I)\cdot\nabla v_h$. Since w is smooth, we do not exclude the point O. From (3.2),

$$
\int_\Omega \nabla(w - w_I)\cdot\nabla v_h = \sum_{\tau\in\mathcal{M}_h} \int_\tau \nabla(w - w_I)\cdot\nabla v_h = J_1 + J_2, \tag{3.19}
$$

where

$$J_j = \sum_{e\in\mathcal{E}_{j,h}} \left[(\beta_e - \beta_e') \int_e \frac{\partial^2 w}{\partial \boldsymbol{t}_e^2} \frac{\partial v_h}{\partial \boldsymbol{t}_e} + (\gamma_e - \gamma_e') \int_e \frac{\partial^2 w}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e} \frac{\partial v_h}{\partial \boldsymbol{t}_e} \right.$$
$$+ \beta_e \int_e \frac{\partial^2 (w_Q - w)}{\partial \boldsymbol{t}_e^2} \frac{\partial v_h}{\partial \boldsymbol{t}_e} + \gamma_e \int_e \frac{\partial^2 (w_Q - w)}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e} \frac{\partial v_h}{\partial \boldsymbol{t}_e}$$
$$\left. + \beta_e' \int_e \frac{\partial^2 (w - w_Q)}{\partial \boldsymbol{t}_e^2} \frac{\partial v_h}{\partial \boldsymbol{t}_e} + \gamma_e' \int_e \frac{\partial^2 (w - w_Q)}{\partial \boldsymbol{t}_e \partial \boldsymbol{n}_e} \frac{\partial v_h}{\partial \boldsymbol{t}_e} \right], \quad j = 1, 2.$$

By a similar argument as for I_1 and I_2, we can prove that

$$\left| \int_\Omega \nabla(w - w_I) \cdot \nabla v_h \right| \lesssim \frac{1}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)} . \tag{3.20}$$

Now, the proof of the lemma follows from (3.12), (3.18), and (3.20). ☐

Applying Lemma 3.3 we obtain the following superconvergence result between the finite element solution u_h and the linear interpolant u_I of the solution of (1.1).

THEOREM 3.4. *Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha, \sigma, \delta/2)$ *with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$ and that $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|\nabla(u_h - u_I)\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \tag{3.21}$$

Proof. Taking $v_h = u_h - u_I$ in Lemma 3.3 we have

$$\|\nabla(u_h - u_I)\|_{L^2(\Omega)}^2 = A(u_h - u_I, v_h) = A(u - u_I, v_h) = \int_\Omega \nabla(u - u_I) \cdot \nabla v_h$$
$$\lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)} = \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}} \|\nabla(u_h - u_I)\|_{L^2(\Omega)} .$$

The proof is completed by canceling $\|\nabla(u_h - u_I)\|_{L^2(\Omega)}$ on both sides of the inequality. ☐

4. Superconvergence between the finite element solution and quadratic interpolation. Most parts of the proof are similar to those for linear elements and therefore are omitted. We emphasize only the differing parts. In this section u_h is the solution of (1.4) with $k = 2$, that is, the quadratic finite element approximation of u.

We first introduce some estimates over triangles from [14]. Recall that $\phi_Q = \Pi_Q \phi$ is the quadratic interpolant defined in (3.1) based on the moment conditions.

LEMMA 4.1. *Assume that $\phi \in H^4(\tau)$; then there holds*

$$\int_\tau \nabla(\phi - \Pi_Q \phi) \cdot \nabla v_h = \sum_{e\subset\partial\tau} \sum_{s=0}^{3} \left(a_e^s(\tau) \frac{|\tau|}{h_e} + b_e^s(\tau) \right) \int_e \frac{\partial^3 \phi}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}} \frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2}$$
$$+ O(h_\tau^3) |\phi|_{H^4(\tau)} \|v_h\|_{H^1(\tau)} \quad \forall v_h \in P_2(\tau), \tag{4.1}$$

where for $s = 0, 1, 2, 3,$

(4.2)

$$|a_e^s(\tau)| + |a_e^s(\tau')| \lesssim h_e^3, \quad |b_e^s(\tau)| + |b_e^s(\tau')| \lesssim h_e^4 \qquad \text{if } e \in \mathcal{E}_h;$$

(4.3)

$$|a_e^s(\tau)|\tau| - a_e^s(\tau')|\tau'|| \lesssim h_e^{5+\alpha}/r_e^{\alpha+\delta(1-\alpha)/3}, |b_e^s(\tau) - b_e^s(\tau')| \lesssim h_e^{4+\alpha}/r_e^{\alpha+\delta(1-\alpha)/3}$$

if $\mathcal{M}_h$ *satisfies* Condition $(\alpha, \sigma, \delta/3)$ *with* $0 < \alpha \le 1$ *and* $0 \le \sigma < 1$*, and if* $e \in \mathcal{E}_{1,h}$*.*

To obtain the superconvergence of $\left\|\nabla(u_h - I_h^2 u)\right\|_{L^2(\Omega)}$, we estimate the difference between two quadratic interpolation operators Π_Q and I_h^2. It is easy to check that [27]

$$\Pi_Q p - I_h^2 p = 0 \quad \forall p \in P_3.$$

By the Bramble–Hilbert lemma, we have

$$\int_\tau (\nabla \Pi_Q \phi - \nabla I_h^2 \phi) \cdot \nabla v_h \lesssim h_\tau^3 \, |\phi|_{H^4(\tau)} \, \|\nabla v_h\|_{L^2(\tau)} \, .$$

Therefore we have the following lemma from (4.1).

Lemma 4.2. *Assume that* $\phi \in H^4(\tau)$*, then there holds*

$$(4.4) \qquad \int_\tau \nabla(\phi - I_h^2 \phi) \cdot \nabla v_h = \sum_{e \subset \partial\tau} \sum_{s=0}^{3} \left(a_e^s(\tau) \frac{|\tau|}{h_e} + b_e^s(\tau) \right) \int_e \frac{\partial^3 \phi}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}} \frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2} + O(h_\tau^3) \, |\phi|_{H^4(\tau)} \, \|v_h\|_{H^1(\tau)} \qquad \text{for } v_h \in P_2(\tau).$$

Recall from Lemma 2.1 that, in the quadratic case, if $h_\tau \eqsim r_\tau^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$, then the total number of degrees of freedom N of the finite element equation (1.4) satisfies

$$(4.5) \qquad N \eqsim \frac{1}{\underline{h}^{2\delta/3}}.$$

The following lemma is analogous to Lemma 3.2. We omit the proof.

Lemma 4.3. *For* v *in decomposition* (1.2)*,*

$$\left\|\nabla v - \nabla I_h^2 v\right\|_{L^2(\tau)} \lesssim h_\tau^\delta \quad \forall \tau \in \mathcal{M}^O.$$

The following lemma is the counterpart of Lemma 3.3 for the quadratic case.

Lemma 4.4. *Assume that* $\mathcal{M}_h$ *satisfies* Condition $(\alpha, \sigma, \delta/3)$ *with* $0 < \alpha \le 1$ *and* $0 \le \sigma < 1$*, and that* $h_\tau \eqsim r_\tau^{1-\delta/3} \underline{h}^{\delta/3}$ *for any* $\tau \in \mathcal{M}_h$*. Then for any* $v_h \in \overset{\circ}{V}{}_h^2$*,*

$$(4.6) \qquad \left| \int_\Omega \nabla(u - I_h^2 u) \cdot \nabla v_h \right| \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1+\rho}} \, \|\nabla v_h\|_{L^2(\Omega)} \, , \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2} \right).$$

Proof. From the decomposition $u = v + w$,

$$(4.7) \qquad \int_\Omega \nabla(u - I_h^2 u) \cdot \nabla v_h = \int_\Omega \nabla(v - I_h^2 v) \cdot \nabla v_h + \int_\Omega \nabla(w - I_h^2 w) \cdot \nabla v_h.$$

We first estimate the term $\int_\Omega \nabla(v - I_h^2 v)\cdot \nabla v_h$. It follows from Lemma 4.2 that

$$\int_\Omega \nabla(v - I_h^2 v)\cdot \nabla v_h = \sum_{\tau\in\mathcal{M}_h}\int_\tau \nabla(v - I_h^2 v)\cdot \nabla v_h = I_1 + I_2 + I_3 + I_4, \tag{4.8}$$

where

$$I_j = \sum_{e=\tau\cap\tau'\in\mathcal{E}_{j,h}\setminus\mathcal{E}^O}\left\{\sum_{s=0}^{3}\left\{\frac{a_e^s(\tau)\,|\tau| - a_e^s(\tau')\,|\tau'|}{h_e} + \left[b_e^s(\tau) - b_e^s(\tau')\right]\right\}\int_e \frac{\partial^3 v}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}}\frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2}\right.$$
$$\left. + O(h_e^3)\,|v|_{H^4(\Omega_e)}\,\|v_h\|_{H^1(\Omega_e)}\right\},\quad j = 1, 2,$$

$$I_3 = \sum_{\tau\in\mathcal{M}^O}\int_\tau \nabla(v - I_h^2 v)\cdot\nabla v_h,$$

$$I_4 = \sum_{e\in\partial\mathcal{E}^O}\left[\sum_{s=0}^{3}\left(a_e^s(\tau)\frac{|\tau|}{h_e} + b_e^s(\tau)\right)\int_e \frac{\partial^3 v}{\partial \boldsymbol{n}_e^s\partial \boldsymbol{t}_e^{3-s}}\frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2} + O(h_\tau^3)\,|v|_{H^4(\tau)}\,\|v_h\|_{H^1(\tau)}\right].$$

Notice that the τ in I_4 is not in $\mathcal{M}^O$.

From Lemma 4.3,

$$|I_3| \lesssim \underline{h}^\delta\,\|\nabla v_h\|_{L^2(\Omega)}. \tag{4.9}$$

It follows from (4.2) and assumption (1.3) that

$$\begin{aligned} |I_4| &\lesssim \sum_{e\in\partial\mathcal{E}^O}\left(h_e^5 r_e^{\delta-3}\,|v_h|_{W^{2,\infty}(\tau)} + h_\tau^3 h_\tau r_e^{\delta-4}\,\|v_h\|_{H^1(\tau)}\right)\\ &\lesssim \sum_{e\in\partial\mathcal{E}^O}\left(h_e^3 r_e^{\delta-3}\,\|v_h\|_{H^1(\tau)}\right) + h_e^4 r_e^{\delta-4}\,\|v_h\|_{H^1(\tau)} \lesssim \underline{h}^\delta\,\|v_h\|_{H^1(\Omega)}. \end{aligned}\tag{4.10}$$

Here we have used the inverse estimate $|v_h|_{W^{2,\infty}(\tau)} \lesssim h_e^{-2}\,\|v_h\|_{H^1(\tau)}$ and the fact that $h_e \eqsim r_e \eqsim \underline{h}$ for $e\in\partial\mathcal{E}^O$.

Next we estimate I_1. It follows from Lemma 4.1 and assumption (1.3) that

$$\begin{aligned} |I_1| &\lesssim \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O}\left[\frac{h_e^{5+\alpha}}{r_e^{\alpha+\delta(1-\alpha)/3}} r_e^{\delta-3}\,|v_h|_{W^{2,\infty}(\tau)} + h_e^3 h_\tau r_\tau^{\delta-4}\,\|v_h\|_{H^1(\Omega_e)}\right]\\ &\lesssim \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O}\left[h_e^{3+\alpha} r_e^{\delta-3-\alpha-\delta(1-\alpha)/3} + h_e^4 r_e^{\delta-4}\right]\|v_h\|_{H^1(\Omega_e)}\\ &\lesssim \left\{\sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O}\left[h_e^2 h_e^{4+2\alpha} r_e^{2\delta-6-2\alpha-2\delta(1-\alpha)/3} + h_e^2 h_e^6 r_e^{2\delta-8}\right]\right\}^{1/2}\|v_h\|_{H^1(\Omega)}\\ &\lesssim \left\{\sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O}\left[h_e^2 \underline{h}^{2\delta(2+\alpha)/3} r_e^{(4+2\alpha)(1-\delta/3)} r_e^{2\delta-6-2\alpha-2\delta(1-\alpha)/3}\right.\right.\\ &\qquad\left.\left. + h_e^2\underline{h}^{2\delta} r_e^{6-2\delta} r_e^{2\delta-8}\right]\right\}^{1/2}\|v_h\|_{H^1(\Omega)}. \end{aligned}$$

Here we have used $h_e \eqsim r_e^{1-\delta/3}\underline{h}^{\delta/3}$ to derive the last inequality. Therefore

(4.11)

$$I_1 \lesssim \left\{ \underline{h}^{2\delta(2+\alpha)/3} \sum_{e\in\mathcal{E}_{1,h}\setminus\mathcal{E}^O} h_e^2 r_e^{-2} \right\}^{1/2} \|v_h\|_{H^1(\Omega)} \lesssim \left\{ \underline{h}^{2\delta(2+\alpha)/3} \sum_{\tau\in\mathcal{M}_h\setminus\mathcal{M}^O} h_\tau^2 r_\tau^{-2} \right\}^{1/2} \times \|v_h\|_{H^1(\Omega)}$$

$$\lesssim \left\{ \underline{h}^{2\delta(2+\alpha)/3} \sum_{\tau\in\mathcal{M}_h\setminus\mathcal{M}^O} \int_\tau r^{-2} \right\}^{1/2} \|v_h\|_{H^1(\Omega)} \lesssim \underline{h}^{\delta(2+\alpha)/3} \left|\ln \underline{h}\right|^{1/2} \|v_h\|_{H^1(\Omega)}.$$

By a similar argument for (3.17) we can show that

$$|I_2| \lesssim \underline{h}^\delta \left\{\#\mathcal{E}_{2,h}\right\}^{1/2} \|v_h\|_{H^1(\Omega)} \lesssim \underline{h}^\delta \left\{N^\sigma\right\}^{1/2} \|v_h\|_{H^1(\Omega)}. \tag{4.12}$$

Notice that $\|v_h\|_{H^1(\Omega)} \lesssim \|\nabla v_h\|_{L^2(\Omega)}$ from Poincaré's inequality. Combining (4.8)–(4.12), we have

$$\begin{aligned} \left|\int_\Omega \nabla(v - v_I)\cdot\nabla v_h\right| &\lesssim \left(\underline{h}^{\delta(2+\alpha)/3}\left|\ln\underline{h}\right|^{1/2} + \underline{h}^\delta\left\{N^\sigma\right\}^{1/2}\right)\|\nabla v_h\|_{L^2(\Omega)} \\ &\lesssim \frac{1+(\ln N)^{1/2}}{N^{1+\rho}}\|\nabla v_h\|_{L^2(\Omega)}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \end{aligned} \tag{4.13}$$

The estimate for the term $\int_\Omega \nabla(w - I_h^2 w)\cdot\nabla v_h$ is similar to (4.13). It follows from Lemma 4.2 that

$$\int_\Omega \nabla(w - I_h^2 w)\cdot\nabla v_h = \sum_{\tau\in\mathcal{M}_h}\int_\tau \nabla(w - I_h^2 w)\cdot\nabla v_h = J_1 + J_2, \tag{4.14}$$

where

$$J_j = \sum_{e=\tau\cap\tau'\in\mathcal{E}_{j,h}} \left\{ \sum_{s=0}^{3}\left\{ \frac{a_e^s(\tau)\,|\tau| - a_e^s(\tau')\,|\tau'|}{h_e} + \left[b_e^s(\tau) - b_e^s(\tau')\right]\right\} \int_e \frac{\partial^3 w}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}} \frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2} + O(h_e^3)\,|w|_{H^4(\Omega_e)}\,\|v_h\|_{H^1(\Omega_e)} \right\}, \quad j = 1, 2.$$

There holds

$$\left|\int_\Omega \nabla(w - w_I)\cdot\nabla v_h\right| \lesssim \frac{1}{N^{1+\rho}}\|\nabla v_h\|_{L^2(\Omega)}. \tag{4.15}$$

Now, the conclusion follows from (4.7), (4.13), and (4.15). ∎

Applying Lemma 4.4 we obtain the following superconvergence result between the quadratic finite element approximation u_h and the quadratic interpolant $I_h^2 u$ of the solution of problem (1.1).

THEOREM 4.5. *Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha, \sigma, \delta/3)$ *with $0 < \alpha \le 1$ and $0 \le \sigma < 1$, and that $h_\tau \eqsim r_\tau^{1-\delta/3}\underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\left\|\nabla(u_h - I_h^2 u)\right\|_{L^2(\Omega)} \lesssim \frac{1+(\ln N)^{1/2}}{N^{1+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \tag{4.16}$$

5. The asymptotically exact a posteriori error estimators.

In this section, we apply a newly developed gradient recovery operator, called polynomial preserving recovery (PPR) [20, 26, 28], to define an a posteriori error estimator. We further prove some superconvergence properties of the recovery operator. As a consequence, the error estimator based on PPR is asymptotically exact under a mesh density assumption.

5.1. The gradient recovery operator G_h and its superconvergence.

Given a node $z \in \mathcal{N}_h$, we select $n \geq m = (k+2)(k+3)/2$ sampling points $z_j \in \mathcal{N}_h$, $j = 1, 2, \ldots, n$, in an element patch ω_z containing z (z is one of z_j) and fit a polynomial of degree $k+1$, in the least squares sense, with values of u_h at those sampling points. In other words, we are looking for $p_{k+1} \in \mathcal{P}_{k+1}$ such that

$$\sum_{j=1}^{n} (p_{k+1} - u_h)^2(z_j) = \min_{q \in \mathcal{P}_{k+1}} \sum_{j=1}^{n} (q - u_h)^2(z_j). \tag{5.1}$$

The recovered gradient at of z is then defined as

$$G_h u_h(z) = (\nabla p_{k+1})(z). \tag{5.2}$$

It was proved in [20] that the above least squares fitting procedure has a unique solution as long as those n sampling points are not on the same conic curve for the linear element. Conditions for higher order elements were given as well. Furthermore, the gradient recovery operator $G_h : C(\Omega) \mapsto V_h^k \times V_h^k$, $k = 1$ or 2, has the following properties:

(i) $\|G_h v_h\|_{L^2(\Omega)} \lesssim \|\nabla v_h\|_{L^2(\Omega)} \quad \forall v_h \in V_h^k$.

(ii) For any nodal point z, $(G_h p)(z) = \nabla p(z)$ if $p \in P_{k+1}(\omega_z)$.

(iii) $|(G_h \phi)(z)| \lesssim \frac{1}{h_\tau} \max_{z' \in \mathcal{N}_h \cap \omega_z} |\phi(z')|$ for any node z in an element $\tau \in \mathcal{M}_h$.

(iv) $G_h \phi = G_h I_h^k \phi$.

Since $I_h^k \phi$ and ϕ have the same nodal values and G_h uses only nodal values, (iv) is clear. The polynomial preserving property (ii) can be established easily by the least squares procedure [28]. A key observation is that G_h provides a finite difference scheme at each node $z \in \mathcal{N}_h$; therefore, (iii) is obvious. Under a very mild mesh condition, "the sum of any two adjacent angles in $\mathcal{M}_h$ is at most π," the boundedness property (i) can be proved, though it is not trivial. The reader is referred to [20, 26, 28] for more details.

We first consider the case of linear finite elements and then state the corresponding results for quadratic elements since the proofs are similar. We have from (i),

$$\begin{aligned} \|G_h u_h - \nabla u\|_{L^2(\Omega)} &\leq \|G_h u_h - G_h u_I\|_{L^2(\Omega)} + \|G_h u_I - \nabla u\|_{L^2(\Omega)} \\ &\lesssim \|\nabla(u_h - u_I)\|_{L^2(\Omega)} + \|G_h u_I - \nabla u\|_{L^2(\Omega)}. \end{aligned} \tag{5.3}$$

Here u_I is the linear interpolant of u. The estimate for the first term of the right hand side of the inequality (5.3) is given in Theorem 3.4. To estimate the second term we need the following lemma.

LEMMA 5.1. *Under properties* (ii)–(iii)*, for any element $\tau \in \mathcal{M}_h$ and any function $\phi \in W^{3,\infty}(\tilde{\tau})$,*

$$\|G_h \phi_I - \nabla \phi\|_{L^2(\tau)} \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde{\tau})},$$

where $\tilde{\tau} = \bigcup \{\omega_z : z \in \mathcal{N}_h \cap \tau\}$ and ϕ_I is the linear interpolant of ϕ.

Proof. Let $(\nabla\phi)_I$ be the linear interpolant of $\nabla\phi$. Then

$$\|G_h\phi_I - \nabla\phi\|_{L^2(\tau)} \le \|G_h\phi_I - (\nabla\phi)_I\|_{L^2(\tau)} + \|(\nabla\phi)_I - \nabla\phi\|_{L^2(\tau)}. \tag{5.4}$$

The standard theory of finite element interpolation estimates says that [6]

$$\|(\nabla\phi)_I - \nabla\phi\|_{L^2(\tau)} \lesssim h_\tau^2 |\phi|_{H^3(\tau)} \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde\tau)}. \tag{5.5}$$

For a node $z \in \tau$, let $\phi_2(x,y)$ be the 2nd-degree Taylor expansion of ϕ at the point z. It is clear that

$$|\phi(x,y) - \phi_2(x,y)| \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde\tau)} \quad \forall (x,y) \in \tilde\tau.$$

By properties (ii) and (iii),

$$\begin{aligned}\left|\big(G_h\phi_I - (\nabla\phi)_I\big)(z)\right| &= \left|\big(G_h\phi_I - \nabla\phi\big)(z)\right| = \left|\big(G_h(\phi_I - \phi_2) - (\nabla\phi - \nabla\phi_2)\big)(z)\right| \\ &= \left|\big(G_h(\phi_I - \phi_2)\big)(z)\right| \lesssim \frac{1}{h_\tau} \max_{z' \in \mathcal{N}_h \cap \omega_z} |(\phi - \phi_2)(z')| \\ &\lesssim h_\tau^2 |\phi|_{W^{3,\infty}(\omega_z)}.\end{aligned}$$

Therefore

$$\|G_h\phi_I - (\nabla\phi)_I\|_{L^2(\tau)} \lesssim h_\tau \max_{z \in \mathcal{N}_h \cap \tau} \left|\big(G_h\phi_I - (\nabla\phi)_I\big)(z)\right| \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde\tau)}. \tag{5.6}$$

The proof of the lemma is completed by combining (5.4)–(5.6). ∎

The following theorem is devoted to the estimate of the second term of (5.3).

THEOREM 5.2. *Assume that $h_\tau \eqsim r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|G_h u_I - \nabla u\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N}. \tag{5.7}$$

Proof. Recall the decomposition (1.2) $u = v + w$, we have, by the triangular inequality,

$$\|G_h u_I - \nabla u\|_{L^2(\Omega)} \le \|G_h v_I - \nabla v\|_{L^2(\Omega)} + \|G_h w_I - \nabla w\|_{L^2(\Omega)}, \tag{5.8}$$

where $v_I = I_h^1 v$ and $w_I = I_h^1 w$ are the linear interpolants of v and w, respectively.

We first estimate the singular part $\|G_h v_I - \nabla v\|_{L^2(\Omega)}$. Introduce the set of triangles $\mathcal{M}^{\tilde O} = \{\tau \in \mathcal{M}_h :$ the origin $O \in \tilde\tau\}$. For any $\tau \in \mathcal{M}^{\tilde O}$,

$$\|G_h v_I - \nabla v\|_{L^2(\tau)} \le \|G_h v_I\|_{L^2(\tau)} + \|\nabla v\|_{L^2(\tau)}. \tag{5.9}$$

By property (ii), $G_h C = 0$ for any constant C. Thus, from property (iii),

$$\begin{aligned}\|G_h v_I\|_{L^2(\tau)} = \|G_h(v_I - v(O))\|_{L^2(\tau)} &\lesssim h_\tau \max_{z \in \mathcal{N}_h \cap \tau} \left|G_h\big(v_I - v(O)\big)(z)\right| \\ &\lesssim h_\tau \frac{1}{h_\tau} \max_{z' \in \mathcal{N}_h \cap \tilde\tau} |v(z') - v(O)| \\ &= \max_{z' \in \mathcal{N}_h \cap \tilde\tau} \left|\int_0^1 \frac{d}{dt} v(z't) \mathrm{d}t\right| = \max_{z' \in \mathcal{N}_h \cap \tilde\tau} \left|\int_0^1 z' \cdot \nabla v(z't) \mathrm{d}t\right|.\end{aligned}$$

Since $\tau \in \mathcal{M}^{\widetilde{O}}$, $|z'| \lesssim \underline{h}$. It follows from assumption (1.3) that

$$\|G_h v_I\|_{L^2(\tau)} \lesssim \int_0^1 \underline{h} \cdot (\underline{h}t)^{\delta-1} \mathrm{d}t \lesssim \underline{h}^\delta. \tag{5.10}$$

On the other hand,

$$\|\nabla v\|_{L^2(\tau)} \lesssim \left(\int_\tau |\nabla v|^2\right)^{1/2} \lesssim \left(\int_\tau r^{2\delta-2}\right)^{1/2} \lesssim \left(\int_0^{c\underline{h}} r^{2\delta-2} r \,\mathrm{d}r\right)^{1/2} \lesssim \underline{h}^\delta. \tag{5.11}$$

Here $c\underline{h}$ is the diameter of $\tilde{\tau}$. Combining (5.9), (5.10), and (5.11), we obtain

$$\|G_h v_I - \nabla v\|_{L^2(\tau)} \lesssim \underline{h}^\delta \quad \text{for } \tau \in \mathcal{M}^{\widetilde{O}}. \tag{5.12}$$

It follows from Lemma 5.1 and (1.3) that

$$\|G_h v_I - \nabla v\|_{L^2(\tau)} \lesssim h_\tau^3 \, |v|_{W^{3,\infty}(\tilde{\tau})} \lesssim h_\tau^3 r_\tau^{\delta-3} \quad \text{for } \tau \in \mathcal{M}_h \setminus \mathcal{M}^{\widetilde{O}}, \tag{5.13}$$

where r_τ is the distance form O to the barycenter of τ. Therefore from $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$,

$$\begin{aligned}
\|G_h v_I - \nabla v\|^2_{L^2(\Omega)} &= \sum_{\tau \in \mathcal{M}_h} \|G_h v_I - \nabla v\|^2_{L^2(\tau)} \lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\widetilde{O}}} h_\tau^6 r_\tau^{2\delta-6} \\
&\lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\widetilde{O}}} h_\tau^2 r_\tau^{4-2\delta} \underline{h}^{2\delta} r_\tau^{2\delta-6} \lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\widetilde{O}}} \underline{h}^{2\delta} h_\tau^2 r_\tau^{-2} \\
&\lesssim \underline{h}^{2\delta} + \underline{h}^{2\delta} \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\widetilde{O}}} \int_\tau r^{-2} \lesssim \underline{h}^{2\delta} + \underline{h}^{2\delta} \int_{\underline{h}}^1 r^{-1} \,\mathrm{d}r \lesssim \underline{h}^{2\delta} \\
&\quad + \underline{h}^{2\delta} \, |\ln \underline{h}| \, .
\end{aligned}$$

Therefore Lemma 2.1 implies that

$$\|G_h v_I - \nabla v\|_{L^2(\Omega)} \lesssim \underline{h}^\delta \big(1 + |\ln \underline{h}|^{1/2}\big) \lesssim \frac{1 + (\ln N)^{1/2}}{N}. \tag{5.14}$$

Next we estimate the term $\|G_h w_I - \nabla w\|_{L^2(\Omega)}$ in (5.8). Since w is smooth, we do not have to divide $\mathcal{M}_h$ into two parts as above. From Lemma 5.1 and assumption (1.3),

(5.15)

$$\begin{aligned}
\|G_h w_I - \nabla w\|_{L^2(\Omega)} &\lesssim \left(\sum_{\tau \in \mathcal{M}_h} \|G_h w_I - \nabla w\|^2_{L^2(\tau)}\right)^{1/2} \lesssim \left(\sum_{\tau \in \mathcal{M}_h} h_\tau^6\right)^{1/2} \\
&\lesssim \left(\sum_{\tau \in \mathcal{M}_h} h_\tau^2 r_\tau^{4-2\delta} \underline{h}^{2\delta}\right)^{1/2} \lesssim \underline{h}^\delta \left(\int_\Omega r^{4-2\delta}\right)^{1/2} \lesssim \underline{h}^\delta \lesssim \frac{1}{N}.
\end{aligned}$$

The proof of the theorem is completed by inserting estimates (5.14) and (5.15) into inequality (5.8). ∎

The following superconvergence result of the gradient recovery operator G_h can be proved by combining (5.3), Theorem 3.4, and Theorem 5.2.

THEOREM 5.3. *Let u_h be the linear finite element approximation of u. Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha, \sigma, \delta/2)$ *with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and that $h_\tau \eqsim r_\tau^{1-\delta/2}\underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|G_h u_h - \nabla u\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \tag{5.16}$$

We remark that the result of Theorem 5.3 is a superconvergence result since the asymptotically optimal convergence rate of $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ is $O(1/N^{1/2})$.

Next we state the results for quadratic finite elements. The following theorem provides the estimate for the gradient recovery operator G_h. The proof is similar to that of Theorem 5.2 and therefore is omitted.

THEOREM 5.4. *Assume that $h_\tau \eqsim r_\tau^{1-\delta/3}\underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\left\|G_h I_h^2 u - \nabla u\right\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{3/2}}. \tag{5.17}$$

The superconvergence of the gradient recovery operator G_h is presented in the following theorem which is parallel to Theorem 5.3.

THEOREM 5.5. *Let u_h be the quadratic finite element approximation of u. Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha, \sigma, \delta/3)$ *with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$ and that $h_\tau \eqsim r_\tau^{1-\delta/3}\underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|G_h u_h - \nabla u\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \tag{5.18}$$

5.2. The a posteriori error estimators. With preparation from the previous subsections, it is now straightforward to prove the asymptotic exactness of error estimators based on the recovery operator G_h. The global error estimator is naturally defined by

$$\eta_h = \|G_h u_h - \nabla u_h\|_{L^2(\Omega)}. \tag{5.19}$$

THEOREM 5.6. *Let u_h be the linear finite element approximation of u. Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha, \sigma, \delta/2)$ *with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and that $h_\tau \eqsim r_\tau^{1-\delta/2}\underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Furthermore, assume that*

$$\frac{1}{N^{1/2}} \lesssim \|\nabla(u - u_h)\|_{L^2(\Omega)}. \tag{5.20}$$

Then

$$\left|\frac{\eta_h}{\|\nabla(u - u_h)\|_{L^2(\Omega)}} - 1\right| \lesssim \frac{1 + (\ln N)^{1/2}}{N^\rho}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \tag{5.21}$$

The following lemma says that $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ is the asymptotically optimal on the mesh $\mathcal{M}_h$ satisfying $h_\tau \eqsim r_\tau^{1-\delta/2}\underline{h}^{\delta/2}$ as the total number of degrees of freedom $N \to \infty$.

LEMMA 5.7. *Let u_h be the linear finite element approximation of u. Assume that $h_\tau \eqsim r_\tau^{1-\delta/2}\underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|\nabla(u-u_I)\|_{L^2(\Omega)} \lesssim \frac{1}{N^{1/2}} \text{ and hence } \|\nabla(u-u_h)\|_{L^2(\Omega)} \lesssim \frac{1}{N^{1/2}}.$$

Proof. Recall that u is decomposed as $u = v + w$ satisfying (1.3). Noticing that

$$\|\nabla(v-v_I)\|_{L^2(\tau)} \lesssim h_\tau |v|_{H^2(\tau)} \lesssim h_\tau^2 r_\tau^{\delta-2} \quad \forall \tau \in \mathcal{M}_h \setminus \mathcal{M}^O,$$

and that

$$\|\nabla(w-w_I)\|_{L^2(\tau)} \lesssim h_\tau |w|_{H^2(\tau)} \lesssim h_\tau^2 \quad \forall \tau \in \mathcal{M}_h,$$

we have, by Lemma 3.2,

$$\begin{aligned}
\|\nabla(u-u_I)\|_{L^2(\Omega)}^2 &\lesssim \|\nabla(v-v_I)\|_{L^2(\Omega)}^2 + \|\nabla(w-w_I)\|_{L^2(\Omega)}^2 \\
&= \sum_{\tau\in\mathcal{M}_h} \left(\|\nabla(v-v_I)\|_{L^2(\tau)}^2 + \|\nabla(w-w_I)\|_{L^2(\tau)}^2 \right) \\
&\lesssim \underline{h}^{2\delta} + \sum_{\tau\in\mathcal{M}_h\setminus\mathcal{M}^O} h_\tau^4 r_\tau^{2\delta-4} \lesssim \underline{h}^{2\delta} + \sum_{\tau\in\mathcal{M}_h\setminus\mathcal{M}^O} h_\tau^2 r_\tau^{2-\delta} \underline{h}^{\delta} r_\tau^{2\delta-4} \\
&\lesssim \underline{h}^{2\delta} + \underline{h}^{\delta} \int_\Omega r^{\delta-2} \lesssim \underline{h}^{2\delta} + \underline{h}^{\delta}.
\end{aligned}$$

In light of Lemma 2.1, we obtain

$$\|\nabla(u-u_I)\|_{L^2(\Omega)}^2 \lesssim \frac{1}{N^2} + \frac{1}{N},$$

which completes the proof of the lemma. ∎

The following lemma says that, for the quadratic finite element approximation u_h, $\|\nabla(u-u_h)\|_{L^2(\Omega)}$ is asymptotically optimal on the mesh $\mathcal{M}_h$ satisfying $h_\tau \eqsim r_\tau^{1-\delta/3}\underline{h}^{\delta/3}$ as the total number of degrees of freedom $N \to \infty$.

LEMMA 5.8. *Let u_h be the quadratic finite element approximation of u. Assume that $h_\tau \eqsim r_\tau^{1-\delta/3}\underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\left\|\nabla(u-I_h^2 u)\right\|_{L^2(\Omega)} \lesssim \frac{1}{N} \text{ and hence } \|\nabla(u-u_h)\|_{L^2(\Omega)} \lesssim \frac{1}{N}.$$

By Theorem 5.5, we can prove the asymptotic exactness of error estimators based on the recovery operator G_h for quadratic elements.

THEOREM 5.9. *Let u_h be the quadratic finite element approximation of u. Assume that $\mathcal{M}_h$ satisfies* Condition $(\alpha,\sigma,\delta/3)$ *with $0<\alpha\le 1$ and $0\le\sigma<1$ and that $h_\tau \eqsim r_\tau^{1-\delta/3}\underline{h}^{\delta/3}$ for any $\tau\in\mathcal{M}_h$. Furthermore, assume that*

$$\frac{1}{N} \lesssim \|\nabla(u-u_h)\|_{L^2(\Omega)}. \tag{5.22}$$

Then

$$\left|\frac{\eta_h}{\|\nabla(u-u_h)\|_{L^2(\Omega)}} - 1\right| \lesssim \frac{1+(\ln N)^{1/2}}{N^\rho}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \tag{5.23}$$

6. Implementation and numerical examples. In this section we present some numerical examples to verify the asymptotic exactness of the error estimator η_h based on the recovery operator G_h using quadratic finite elements. For examples on linear elements we refer to [12].

Implementation of the adaptive algorithm in this section is based on FEMLAB.[1] We define the local a posteriori error estimator on element τ as

$$\eta_\tau = \|G_h u_h - \nabla u_h\|_{L^2(\tau)},$$

and the global error estimator as

$$\eta_h = \left(\sum_{\tau \in \mathcal{M}_h} \eta_\tau^2 \right)^{1/2}.$$

Now we describe the adaptive algorithm used in this paper.

Algorithm. Given the tolerance $\texttt{TOL} > 0$,

- generate an initial mesh $\mathcal{M}_h$ over Ω;
- while $\eta_h > \texttt{TOL}$ do
  - choose a set of elements $\widehat{\mathcal{M}}_h \subset \mathcal{M}_h$ such that

$$\left(\sum_{\tau \in \widehat{\mathcal{M}}_h} \eta_\tau^2 \right)^{1/2} > 0.7 \left(\sum_{\tau \in \mathcal{M}_h} \eta_\tau^2 \right)^{1/2},$$

    then refine the elements in $\widehat{\mathcal{M}}_h$. Update the mesh $\mathcal{M}_h$.
  - solve the discrete problem (1.4) on $\mathcal{M}_h$.
  - compute error estimators on $\mathcal{M}_h$.

  end while

Remark 6.1. The marking strategy, that is, the method of how to choose $\widehat{\mathcal{M}}_h$ for refinements used in our algorithm, is well known in the adaptive finite element community. Actually, it was used, e.g., in [9, 18] to design convergent finite element algorithms. In our implementation of the algorithm, the elements in $\widehat{\mathcal{M}}_h$ are chosen from the elements which have larger local a posteriori error estimators η_τ.

Example 1. The Laplace equation on the L-shaped domain of Figure 6.1 with the Dirichlet boundary condition is chosen so that the true solution is $r^{2/3}\sin(2\theta/3)$ in polar coordinates.

Figure 6.1 plots the initial mesh and the adaptively refined mesh of 3565 elements after 15 adaptive iterations. Figure 6.2 demonstrates asymptotic exactness of the error estimator $\eta_h = \|G_h u_h - \nabla u_h\|_{L^2(\Omega)}$ for the Laplace equation on the L-shaped domain. We see that

$$\|\nabla u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1}), \quad \|G_h u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1.2}),$$

and

$$\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} \Big/ \|\nabla u - \nabla u_h\|_{L^2(\Omega)} \approx 1 + O(N^{-0.5}).$$

[1] http://ecs.rutgers.edu/eitlab/femlab.php.

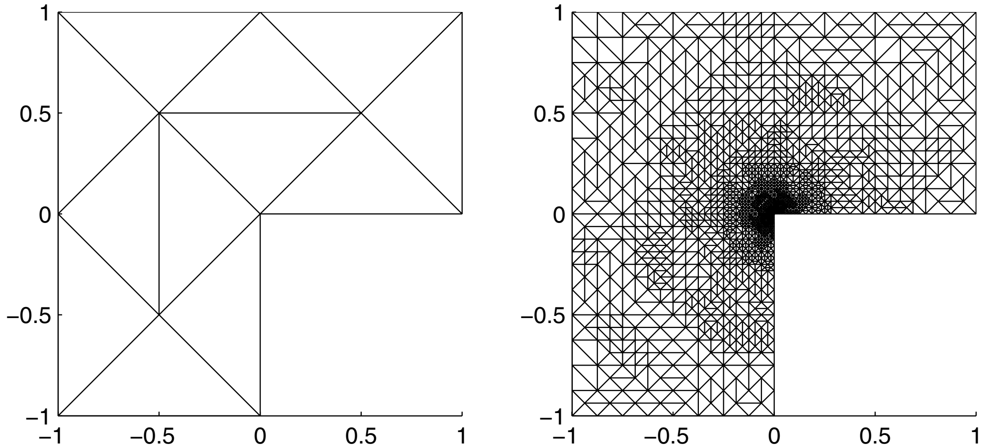

FIG. 6.1. *The initial mesh (left) and the adaptively refined mesh (right) of* 3565 *elements after* 15 *adaptive iterations for the Laplace equation on the L-shaped domain.*

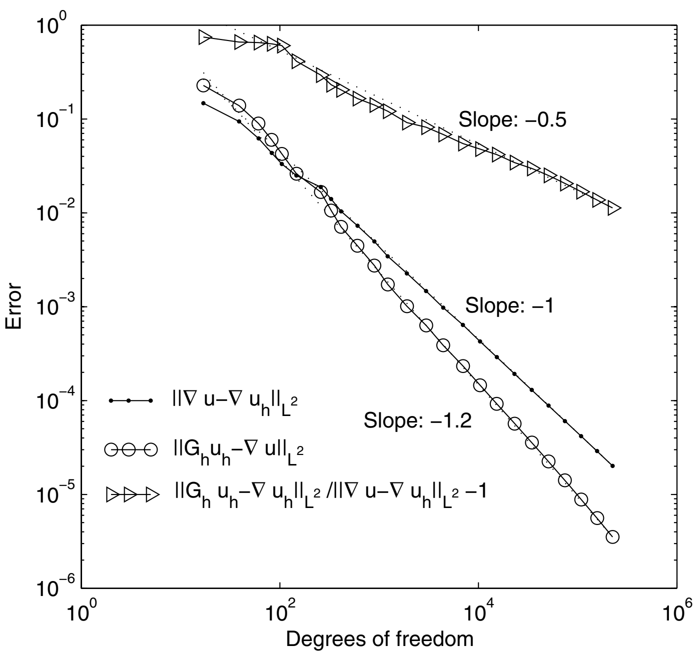

FIG. 6.2. $\|\nabla u - \nabla u_h\|_{L^2(\Omega)}$, $\|\nabla u - G_h u_h\|_{L^2(\Omega)}$, *and* $\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} / \|\nabla u - \nabla u_h\|_{L^2(\Omega)}$ -1 *versus the total number of degrees of freedom for the Laplace equation on the L-shaped domain. Dotted lines give reference slopes.*

Notice that the decay of $\|\nabla u_h - \nabla u\|_{L^2(\Omega)}$ is quasi-optimal, $\|G_h u_h - \nabla u\|_{L^2(\Omega)}$ is superconvergent with order $O(N^{-1.2})$, and $\eta_h / \|\nabla u - \nabla u_h\|_{L^2(\Omega)}$ approaches 1 at the rate of $O(N^{-0.5})$. In this paper, the L^2 norms are calculated by the six points Gauss quadrature rule over triangles.

Let us have a close look at the mesh density assumption $h_\tau \eqsim r_\tau^{1-\delta/3} \underline{h}^{\delta/3} = r_\tau^{7/9} \underline{h}^{2/9}$ for $\delta = 2/3$. We shall verify this on the final mesh, which has 112880 elements after 24 adaptive iterations. We choose $\underline{h} = \min_{\tau\in\mathcal{M}_h} h_\tau \approx 5.96 \times 10^{-8}$ and have

$$0.44 \leq \frac{h_\tau}{r_\tau^{7/9}\underline{h}^{2/9}} \leq 2.35$$

for all elements $\tau \in \mathcal{M}_h$. Note that the ratio between the upper and lower bounds is less than 6. This fact indicates that all elements in the final mesh satisfy the mesh density assumption.

Next, let us examine the condition (α, σ, μ) on the final mesh. Here $\mu = \delta/3 = 2/9$. It is shown that, for every $e \in \mathcal{E}_h$, Ω_e is a $3.92 \times h_e^{1+0.4}/r_e^{0.4+\mu(1-0.4)}$ approximate parallelogram. That is, the final mesh satisfies Condition $(0.4, 0, 2/9)$.

Example 2. Let $\Omega = \{(x_1, x_2) : |x_1|, |x_2| < 0.5\}\backslash\{(x_1, 0) : 0 \leq x_1 < 0.5\}$ be the domain with a crack. We consider the Poisson equation

$$-\Delta u = 1$$

with a Dirichlet boundary condition chosen so that the true solution is $r^{1/2}\sin(\theta/2) - \frac{1}{4}r^2$ in polar coordinates.

Figure 6.3 plots the initial mesh and the adaptively refined mesh of 3353 elements after 16 adaptive iterations. Figure 6.4 shows asymptotic exactness of the error estimator $\eta_h = \|G_h u_h - \nabla u_h\|_{L^2(\Omega)}$ for the crack problem. We see that

$$\|\nabla u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1}), \quad \|G_h u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1.1}),$$

and

$$\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} \Big/ \|\nabla u - \nabla u_h\|_{L^2(\Omega)} \approx 1 + O(N^{-0.3}).$$

Notice that the decay of $\|\nabla u_h - \nabla u\|_{L^2(\Omega)}$ is quasi-optimal, $\|G_h u_h - \nabla u\|_{L^2(\Omega)}$ is superconvergent at an order $O(N^{-1.1})$, and $\eta_h / \|\nabla u - \nabla u_h\|_{L^2(\Omega)}$ approaches 1 at the rate of $O(N^{-0.3})$.

Let us take a close look at the mesh density assumption $h_\tau \eqsim r_\tau^{1-\delta/3}\underline{h}^{\delta/3} = r_\tau^{5/6}\underline{h}^{1/6}$ for $\delta = 1/2$. We verify this on the final mesh, which has 110563 elements after 27 adaptive iterations. We choose $\underline{h} = \min_{\tau\in\mathcal{M}_h} h_\tau \approx 3.67 \times 10^{-9}$ and have

$$0.32 < \frac{h_\tau}{r_\tau^{5/6}\underline{h}^{1/6}} < 1.92$$

for all elements $\tau \in \mathcal{M}_h$. Note that the ratio between the upper and lower bounds is 6. This fact indicates that all elements in the final mesh satisfy the mesh density assumption.

Next, let us examine the condition (α, σ, μ) on the final mesh. Here $\mu = \delta/3 = 1/6$. It is shown that, for every $e \in \mathcal{E}_h$, Ω_e is a $1.49 \times h_e^{1+0.2}/r_e^{0.2+\mu(1-0.2)}$ approximate parallelogram. That is, the final mesh satisfies Condition $(0.2, 0, 1/6)$.

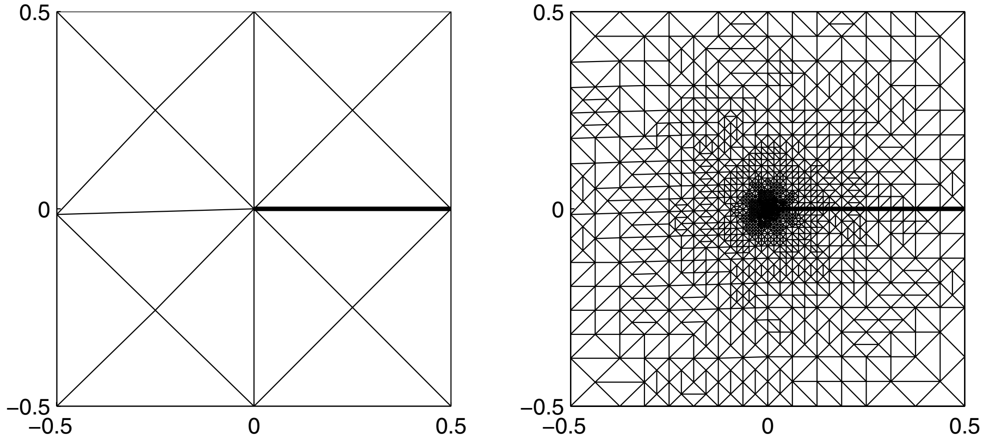

FIG. 6.3. *The initial mesh (left) and the adaptively refined mesh (right) of* 3353 *elements after* 16 *adaptive iterations for the crack problem.*

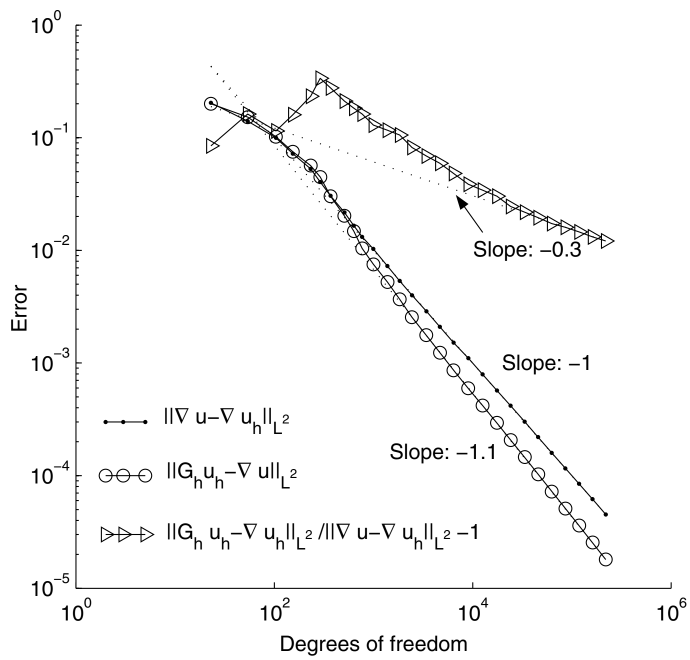

FIG. 6.4. $\|\nabla u - \nabla u_h\|_{L^2(\Omega)}$, $\|\nabla u - G_h u_h\|_{L^2(\Omega)}$, *and* $\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} / \|\nabla u - \nabla u_h\|_{L^2(\Omega)}$ -1 *versus the total number of degrees of freedom for the crack problem. Dotted lines give reference slopes.*

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