A new $P_1$ weak Galerkin method for the Biharmonic equation

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A new weak Galerkin (WG) finite element method is designed featuring in using the first order polynomial to approximate solution of biharmonic equation. The proposed $P_1$ WG method achieves $O(h)$ convergence in energy norm and $O(h^2)$ in $L^2$ norm in solving the biharmonic equation. This is not possible for the traditional finite element method as the minimum polynomial degree is 2 in order to approximate the biharmonic equation. Numerical tests on various polygonal meshes verify the theory.

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1. Introduction

This paper is concerned with a $P_1$ weak Galerkin finite element method for solving the biharmonic equation with boundary conditions: Seeking an unknown function $u(x)$ satisfying

$$\Delta^2 u = f, \quad \text{in } \Omega, \quad (1.1)$$
$$u = \zeta, \quad \text{on } \partial \Omega, \quad (1.2)$$
$$\frac{\partial u}{\partial n} = \phi, \quad \text{on } \partial \Omega, \quad (1.3)$$

where $\Delta$ is the Laplacian operator, $\zeta = \zeta(x)$ and $\phi = \phi(x)$ are given functions defined on the boundary of the domain $\Omega$. Assume that $\Omega \in \mathbb{R}^d$ with $d = 2, 3$ is a bounded and convex domain with smooth boundary $\partial \Omega$.

The biharmonic equation models a plate bending problem. The mathematical theory of the finite element method starts from solving this application problem, cf. [1–4]. The standard finite element method, i.e., the conforming element, requires a $C^1$ function space of piecewise polynomials. This would require a high polynomial degree [1,5–7], or a macro-element [2,3,8–12], or a constraint element (where the polynomial degree is reduced at inter-element boundary) [4,13,14].

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Mixed methods for the biharmonic equation avoid using \( C^1 \) elements by reducing the fourth order equation to a system of two second order equations, \([15–18]\). Using nonconforming methods is another way to avoid \( C^1 \) elements, and the Morley element \([19–21]\) is a well known example for its simplicity. In addition, interior penalty discontinuous Galerkin (IPDG) methods and hybridizable discontinuous Galerkin (HDG) method are studied in \([22–25]\).

Similar to the discontinuous Galerkin method, the weak Galerkin finite element method uses discontinuous polynomials on general polyhedral grid. A uniform way of defining weak derivatives makes the weak Galerkin method a simple and efficient method for complicated problems and high order differential equations. For example, it is extremely difficult to construct 3D conforming and nonconforming biharmonic finite elements on tetrahedral grids. But the construction of weak Galerkin finite elements is uniform, independent of grid type, polynomial degree, and space dimension.

There have been some recent attempts on WG methods for the biharmonic equation \([26–29]\). In addition to usual \( u_0 \) and \( u_b \) representing interior and boundary unknowns, another independent unknown \( u_e \) is introduced as a counterpart of the normal derivative of \( u \) in all these works. The main difference is in the selection of piecewise polynomial spaces for the triple \((u_0, u_b, u_e)\): \([26,27]\) used \((P_2, P_2, P_1)\), \([29]\) constructed \((P_2, P_1, P_1)\) triple, and \([28]\) studied \((P_2, P_0, P_0)\) triple. A different approach in \([30]\) used \((P_1, P_1, P_1)\) weak Galerkin triple, and \([28]\) designed a \((P_2, P_2, P_2)\) method recently where the optimal order, the third order \( L^2 \)-convergence is obtained.

In recent years, many finite element schemes have been developed for solving fourth order problems using discontinuous approximations such as IPDG, HDG and WG methods as mentioned above to avoid construction of \( C^1 \) conforming element. Among these discontinuous finite element methods, IPDG and WG methods are in primal form with a symmetric and positive definite system. Comparing with the IPDG methods, the WG methods introduce additional degrees of freedom on element boundary to make the WG method more local and to avoid tuning penalty parameters. The weak form of the biharmonic problem \((1.1)\) is to find \( u \in H^2(\Omega) \) satisfying \( u|_{\partial \Omega} = g \) and \( \frac{\partial u}{\partial n}|_{\partial \Omega} = \phi \) such that

\[
(\Delta u, v) = (f, v) \quad \forall v \in H^2_0(\Omega).
\]  

Since \( \Delta \) is involved in the primal weak form \((1.4)\), it is natural for all the finite element methods referred above in primal form to use \( P_1 \) or higher degree polynomials as approximation. In this paper, we will be the first to try to use \( P_1 \) polynomial in a primal finite element formulation starting from \((1.4)\). As one knows \( \Delta = 0 \) if \( v \) is a first order polynomial, however flexibility of weak Laplacian makes the \( P_1 \) choice possible in a WG formulation. Another method allows to use \( P_1 \) polynomial is the HDG method in \([23]\). However the HDG method is a saddle point problem by rewriting a fourth order PDE to a system of four first order PDEs. In other words, there are no second derivatives involving in the HDG formulation.

A \((P_1, P_1, P_1)\) weak Galerkin finite element method is introduced in this paper. We obtain an optimal order convergence finite element method for the biharmonic equation. Indeed, we achieve the same approximation power as that for solving the second order problems, using piecewise \( P_1 \) polynomials. This is not possible for the traditional finite elements as the minimum polynomial degree is 2 for the biharmonic equation \([10, 20]\). However, with additional gradient/hessian recovery techniques, \( P_1 \) finite elements can still be applied to the biharmonic equation \([32–34]\). But unlike our new weak Galerkin method, these methods would not achieve the optimal order of convergence, at least, in the \( L^2 \)-norm. Numerical tests on different polygonal meshes confirm the theory with the optimal order of convergence.

2. Weak Galerkin finite element schemes

Let \( \mathcal{T}_h \) be a partition of the domain \( \Omega \) into polygons in 2D or polyhedra in 3D. Assume that \( \mathcal{T}_h \) is shape regular in the sense as defined in \([35]\). We denote by \( h_T \) its diameter and mesh size \( h = \max_{T \in \mathcal{T}_h} h_T \) for \( \mathcal{T}_h \).

We define two finite element spaces \( V_h \) and \( V_h^0 \) as follows,

\[
V_h = \{ v = \{v_0, v_b, v_g\} \in P_1(T) \times P_1(e) \times [P_1(e)]^d, e \subset \partial T, \ T \in \mathcal{T}_h \},
\]

and

\[
V_h^0 = \{ v \in V_h : v_b = 0, \ v_g \cdot n = 0, \text{ on } \partial \Omega \}.
\]

where \( n \) is the unit outward normal of \( \partial \Omega \).

For any \( v = \{v_0, v_b, v_g\} \), a weak Laplacian \( \Delta_w v \in P_1(T) \) is defined on \( T \) by

\[
(\Delta_w v, \phi)_T = (v_0, \Delta \phi)_T - (v_b, \nabla \phi \cdot n)_{\partial T} + (v_g \cdot n, \phi)_{\partial T}, \quad \forall \phi \in P_1(T),
\]

where \( n \) is the unit outward normal of \( \partial T \).

Let \( Q_0, Q_b \) and \( Q_g \) be the locally defined \( L^2 \) projections onto \( P_1(T) \), \( P_1(e) \) and \( [P_1(e)]^d \) accordingly for each element \( T \in \mathcal{T}_h \) and \( e \subset \partial T \). For the true solution \( u \) of \((1.1)–(1.3)\), we define \( Q_0 u \) as

\[
Q_0 u = [Q_0 u, Q_b u, Q_g (\nabla u)] \in V_h.
\]

First, we introduce some notations,

\[
(v, w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T v w \, dx.
\]
Proof. For any \( v = \{w_0, v_b, \mathbf{v}_g\} \) and \( v = \{v_0, v_b, \mathbf{v}_g\} \) in \( V_h \), we introduce a bilinear form as follows
\[
\mathcal{S}(v, w) = \sum_{T \in \mathcal{T}_h} h_T (\nabla w_0 - \mathbf{w}_g, \nabla v_0 - \mathbf{v}_g)_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-1} (w_0 - u_b, v_0 - v_b)_{\partial T}.
\]

**Weak Galerkin Algorithm 1.** A numerical approximation for (1.1)-(1.3) can be obtained by seeking \( u_h = \{u_0, u_b, u_g\} \in V_h \) satisfying \( u_h = Q_h f \) and \( u_g \cdot n = Q_h \phi \) on \( \partial \Omega \) and the following equation:
\[
(\Delta u_h, \Delta u) + \mathcal{S}(u_h, v) = (f, v_0), \quad \forall \ v = \{v_0, v_b, \mathbf{v}_g\} \in V_h^0.
\] (2.2)

The following lemma implies the well-posedness of the WG finite element formulation (2.2).

**Lemma 2.1.** For any \( v \in V_h^0 \), let \( \|v\| \) be given as follows
\[
\|v\|^2 = (\Delta v, \Delta v) + \mathcal{S}(v, v).
\] (2.3)

Then, \( \|\cdot\| \) defines a norm in \( V_h^0 \).

**Proof.** We only need to verify the positivity for the semi-norm \( \|v\| \). Assume that \( \|v\| = 0 \) for \( \{v_0, v_b, \mathbf{v}_g\} \in V_h^0 \). It implies that \( \Delta v = 0 \) in each element \( T \), and \( v_0 = v_b \), and \( \nabla v_0 = \mathbf{v}_g \) on \( \partial T \).

Since \( v_0 \in P_1(T) \), we have \( \Delta v_0 = 0 \) on each element \( T \). Together with \( v_0 = v_b \) and \( \nabla v_0 = \mathbf{v}_g \) on \( \partial T \), we have that \( v \) is a smooth harmonic function on \( \Omega \). The boundary conditions of \( v_b = 0 \) and \( \mathbf{v}_g \cdot n = 0 \) imply that \( v \equiv 0 \) on \( \Omega \), which completes the proof. \( \square \)

The following commutative property for the Laplacian \( \Delta \) and the weak Laplacian \( \Delta_w \) plays an important role in error analysis.

**Lemma 2.2.** On each element \( T \in \mathcal{T}_h \), we have for any \( v \in H^2(T) \)
\[
\Delta_w(Q_h v) = Q_h(\Delta v).
\] (2.4)

**Proof.** For any \( \tau \in P_1(T) \), we have that
\[
(\Delta_w Q_h v, \tau)_T = (Q_h v, \Delta \tau)_T + (Q_h (\nabla v) \cdot n, \tau)_{\partial T} - (Q_h v, \nabla \tau \cdot n)_{\partial T}
\]
\[
= (\nabla v \cdot n, \tau)_{\partial T} - (v, \nabla \tau \cdot n)_{\partial T}
\]
\[
= (\Delta v, \tau)_T = (Q_h \Delta v, \tau)_T,
\]
which implies the identity (2.4). \( \square \)

For the solution \( u \) of the problem (1.1)-(1.3), we define \( \Delta_w u \) on \( T \in \mathcal{T}_h \) by
\[
(\Delta_w u, \phi)_T = (u, \Delta \phi)_T - (u, \nabla \phi \cdot n)_{\partial T} + (\nabla u \cdot n, \phi)_{\partial T}, \quad \forall \phi \in P_1(T).
\]
Integration by parts gives
\[
\Delta_w u = Q_h \Delta u.
\] (2.5)

3. **Error estimates**

The goal of this section is to establish error estimates for the WG finite element solution \( u_h \) of (2.2). For a given \( T \in \mathcal{T}_h \) with edge \( e \) and a function \( \phi \in H^1(T) \), the following trace inequality holds true (see [35] for details):
\[
\|\phi\|_{e}^2 \leq C (h_T^{-1} \|\phi\|_T^2 + h_T \|\nabla \phi\|_T^2),
\] (3.1)
where \( \|\phi\|_e^2 = \int_e \phi^2 ds \) and \( \|\phi\|_T^2 = \int_T \phi^2 dA \).

**3.1. Error equation**

First, define an error function between the finite element solution and the \( L^2 \) projection of the exact solution as follows,
\[
e_h = \{e_0, e_b, e_g\} = Q_h u - u_h = \{Q_h u - u_0, Q_h u - u_b, Q_g (\nabla u) - u_g\}.
\]
We will develop an equation satisfied by \( e_h \) in the following lemma.
**Lemma 3.1.** Let $u$ and $u_h = \{u_h, u_h, u_h\} \in V_h$ be the solution of (1.1)–(1.3) and (2.2) respectively. Then we have

\[
(\Delta_w e_h, \Delta_w v)_{\mathcal{T}_h} + s(e_h, v) = \ell_1(u, v) - \ell_2(u, v) + s(Q_h u, v), \quad \forall v \in V_h^0,
\]

where

\[
\ell_1(u, v) = (\Delta u - Q_0 \Delta u, (\nabla v_0 - v_0) \cdot n)_{\partial \mathcal{T}_h},
\]

\[
\ell_2(u, v) = (\nabla (\Delta u - Q_0 \Delta u) \cdot n, v_0 - v_0)_{\partial \mathcal{T}_h}.
\]

**Proof.** Using (2.1), integration by parts, and (2.4), we have

\[
(\Delta_w Q_h u, \Delta_w v)_T = (v_0, \Delta (\Delta_w Q_h u)) + (\nabla_T h \cdot u, \Delta_w Q_h u)_{\partial T} - (v_0, \nabla (\Delta_w Q_h u) \cdot n)_{\partial T}
\]

\[
= (\Delta w, v_0)_T + (v_0, \nabla (\Delta_w Q_h u) \cdot n)_{\partial T} - (\nabla v_0, \nabla (\Delta_w Q_h u) \cdot n)_{\partial T}
\]

\[
+ (\nabla_T h \cdot u, \Delta_w Q_h u)_{\partial T} - (v_0, \nabla (\Delta_w Q_h u) \cdot n)_{\partial T}
\]

It follows from (2.4) and summing the above equation over $T \in \mathcal{T}_h$,

\[
(\Delta_w Q_h u, \Delta_w v)_{\mathcal{T}_h} = (\Delta u, v_0)_{\mathcal{T}_h} + (v_0 - v_0, \nabla (\Delta_w Q_h u) \cdot n)_{\partial \mathcal{T}_h}
\]

\[
- (\nabla v_0, \nabla (\Delta_w Q_h u) \cdot n)_{\partial \mathcal{T}_h}.
\]

Testing (1.1) by $v_0$ of $v = \{v_h, v_h, v_h\} \in V_h^0$, we arrive at

\[
(\Delta^2 u, v_0) = (f, v_0).
\]

Using the integration by parts and the facts that $u$ is smooth, and $v_h \cdot n$ and $v_h$ vanishes on $\partial \Omega$, we have

\[
(\Delta^2 u, v_0) = (\Delta u, v_0)_{\mathcal{T}_h} + (\Delta v_0, \nabla (\Delta u) \cdot n)_{\partial \mathcal{T}_h}
\]

\[
= (\Delta u, v_0)_{\mathcal{T}_h} + (\nabla (\Delta u) \cdot n, v_0 - v_0)_{\partial \mathcal{T}_h}.
\]

Combining the above equation with (3.3) leads to

\[
(\Delta^2 u, v_0) = (\Delta w, v_0)_{\mathcal{T}_h} + (\nabla (\Delta u - Q_0 \Delta u) \cdot n, v_0 - v_0)_{\partial \mathcal{T}_h}
\]

which implies

\[
(\Delta_w Q_h u, \Delta_w v)_{\mathcal{T}_h} = (f, v_0) + \ell_1(u, v) - \ell_2(u, v).
\]

Adding $s(Q_h u, v)$ to both sides of the above equation gives

\[
(\Delta_w Q_h u, \Delta_w v)_{\mathcal{T}_h} + s(Q_h u, v) = (f, v_0) + \ell_1(u, v) - \ell_2(u, v) + s(Q_h u, v).
\]

Subtracting (2.2) from (3.4) yields the following error equation

\[
(\Delta_w e_h, \Delta_w v)_{\mathcal{T}_h} + s(e_h, v) = \ell_1(u, v) - \ell_2(u, v) + s(Q_h u, v), \quad \forall v \in V_h^0.
\]

This completes the proof of the lemma. \qed

**3.2. Error estimate in $H^2$ equivalent norm**

The following is an estimate for the error function $e_h$ in the triple-bar norm $\| \cdot \|$.

**Theorem 3.2.** Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (2.2). Assume $u \in H^4(\Omega)$. Then, there exists a constant $C$ such that

\[
\| u_h - u \| \leq Ch \| u \|_4.
\]

**Proof.** By the triangle inequality,

\[
\| u_h - u \| \leq \| u_h - Q_h u \| + \| Q_h u - u \|.
\]

We will estimate $\| u_h - Q_h u \|$ first. By letting $v = e_h$ in the error Eq. (3.2), we obtain the following equation,

\[
\| e_h \|^2 = \ell_1(u, e_h) - \ell_2(u, e_h) + s(Q_h u, e_h).
\]

\[
\| e_h \|^2 = \ell_1(u, e_h) - \ell_2(u, e_h) + s(Q_h u, e_h).
\]
Using the Cauchy–Schwarz inequality and the trace inequality (3.1), we have

\[
\ell_1(u, e_h) = \left| \sum_{T \in T_h} (\Delta u - Q_0 \Delta u, (\nabla e_0 - e_g) \cdot n)_{\partial T} \right| \tag{3.8}
\]

\[
\leq \left( \sum_{T \in T_h} h_T^{-1} \| \Delta u - Q_0 \Delta u \|^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T \| \nabla e_0 - e_g \|^2_{\partial T} \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \sum_{T \in T_h} \left( h_T^{-2} \| \Delta u - Q_0 \Delta u \|^2 + \| \nabla (\Delta u - Q_0 \Delta u) \|^2 \right) \right)^{\frac{1}{2}}
\]

\[
\leq C h \| u \|_4 \| e_h \|,
\]

and

\[
\ell_2(u, e_h) = \left| \sum_{T \in T_h} (\nabla (\Delta u - Q_0 \Delta u) \cdot n, e_0 - e_h)_{\partial T} \right| \tag{3.9}
\]

\[
\leq \left( \sum_{T \in T_h} h_T \| \nabla (\Delta u - Q_0 \Delta u) \|^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \| e_0 - e_h \|^2_{\partial T} \right)^{\frac{1}{2}}
\]

\[
\leq C h \| u \|_4 \| e_h \|.
\]

Similarly, it follows from the Cauchy–Schwarz and (3.1) that

\[
\left| \sum_{T \in T_h} h_T \langle \nabla Q_0 u - Q_0 \nabla u, \nabla e_0 - e_g \rangle_{\partial T} \right| \leq C h \| u \|_2 \| e_h \|, \tag{3.10}
\]

and

\[
\left| \sum_{T \in T_h} h_T^{-1} \langle Q_0 u - Q_0 u, e_0 - e_h \rangle_{\partial T} \right| \leq C h \| u \|_2 \| e_h \|. \tag{3.11}
\]

Substituting (3.8)–(3.11) into (3.7) yields

\[
\| e_h \|^2 \leq C h \| u \|_4 \| e_h \|. \tag{3.12}
\]

Now we estimate \( \| u - Q_0 u \| \). Eqs. (2.4) and (2.5) imply \( \Delta_w(u - Q_0 u) = 0 \). It follows from the definition of \( \| \cdot \| \) that

\[
\| u - Q_0 u \|^2 = s(u - Q_0 u, u - Q_0 u) = s(Q_0 u, Q_0 u). \tag{3.13}
\]

Similar to (3.10) and (3.11), we have

\[
s(Q_0 u, Q_0 u) = \sum_{T \in T_h} h_T \langle \nabla Q_0 u - Q_0 \nabla u, \nabla Q_0 u - Q_0 \nabla u \rangle_{\partial T}
\]

\[
+ \sum_{T \in T_h} h_T^{-1} \langle Q_0 u - Q_0 u, Q_0 u - Q_0 u \rangle_{\partial T}
\]

\[
\leq C h^2 \| u \|^4_4.
\]

Thus, we have

\[
\| u - Q_0 u \| \leq C h \| u \|_4.
\]

This completes the proof of the theorem. \( \square \)

3.3. Error estimate in \( L^2 \) norm

We will establish an estimate for the error function \( e_h \) in the standard \( L^2 \) norm. Consider the following dual problem

\[
\Delta^2 w = e_0 \quad \text{in} \quad \Omega, \tag{3.14}
\]
\[ w = 0, \quad \text{on } \partial \Omega, \quad (3.15) \]
\[ \nabla w \cdot n = 0 \quad \text{on } \partial \Omega. \quad (3.16) \]

Assume that the above dual problem has the following regularity estimate
\[ \|w\|_4 \leq C\|e_0\|. \quad (3.17) \]

**Theorem 3.3.** Let \( u_0 \in V_h \) be the weak Galerkin finite element solution arising from (2.2). Assume \( u \in H^4(\Omega) \) and the dual problem (3.14)–(3.16) has the \( H^4 \) regularity. Then, there exists a constant \( C \) such that
\[ \|u - u_0\| \leq Ch^2\|u\|_4. \quad (3.18) \]

**Proof.** By the triangle inequality,
\[ \|u - u_0\| \leq \|u_0 - Q_h u\| + \|Q_h u - u\|. \quad (3.19) \]

We will estimate \( \|u_0 - Q_h u\| \) first. Testing (3.14) with the error function \( e_0 \) on each element and then using integration by parts to obtain
\[ \|e_0\|^2 = (\Delta^2 w, e_0) = (\nabla(\Delta w) \cdot n, e_0)_{\partial \Omega} - (\Delta w, \nabla e_0 \cdot n)_{\partial \Omega}. \]

Combining the above two equations we obtain
\[ \|e_0\|^2 = (\Delta w, Q_h u, \Delta w, e_h) + \ell_2(w, e_h) - \ell_1(w, e_h). \quad (3.20) \]

Each of the six terms on the right-hand side of (3.20) can be bounded by using the Cauchy–Schwarz inequality and the trace inequality (3.1) as follows.

For the first term, we have
\[ \ell_1(u, Q_h w) = \left| \sum_{T \in T_h} \langle \Delta u - Q_h \Delta u, (\nabla Q_h w - Q_h \nabla w) \cdot n \rangle_{\partial T} \right| \]
\[ \leq \left( \sum_{T \in T_h} h_T^{-1} \|\Delta u - Q_h \Delta u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^2 \|\nabla Q_h w - \nabla w\|_{\partial T}^2 \right)^{\frac{1}{2}} \]
\[ \leq Ch^2\|u\|_4\|w\|_3. \quad (3.21) \]

We have for the second term
\[ \left| \sum_{T \in T_h} \langle \nabla(\Delta u - Q_h \Delta u) \cdot n, Q_h w - Q_h w \rangle_{\partial T} \right| \]
\[ \leq \left( \sum_{T \in T_h} h_T \|\nabla(\Delta u - Q_h \Delta u)\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \|Q_h w - w\|_{\partial T}^2 \right)^{\frac{1}{2}} \]
\[ \leq Ch^2\|u\|_4\|w\|_3, \quad (3.22) \]

and for the third term
\[ \left| s(Q_h u, Q_h w) \right| \]
\[ \leq \left| \sum_{T \in T_h} h_T \langle \nabla Q_h u - Q_h \nabla u, \nabla Q_h w - Q_h \nabla w \rangle_{\partial T} \right| \]
\[ \leq Ch^2\|u\|_4\|w\|_3. \quad (3.23) \]
\begin{align*}
+ & \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} (Q_0 u - Q_0 u, Q_0 w - Q_0 w) \right|_T \\
\leq & \left( \sum_{T \in \mathcal{T}_h} h_T \| \nabla Q_0 u - \nabla u \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T \| \nabla Q_0 w - \nabla w \|_{\partial T}^2 \right)^{1/2} \\
+ & \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \| Q_0 u - u \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \| Q_0 w - w \|_{\partial T}^2 \right)^{1/2} \\
\leq & Ch^2 ||u||_2 ||w||_2.
\end{align*}

Similar to (3.23), we have

\begin{equation}
| s(e_h, Q_0 w) | \leq \sum_{T \in \mathcal{T}_h} h_T \langle \nabla e_0 - e_g, \nabla Q_0 w - Q_g \nabla w \rangle_{\partial T}
\end{equation}

\begin{equation}
+ \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle e_0 - e_b, Q_0 w - Q_b w \rangle_{\partial T} \leq Ch \| e_h \| ||w||_2.
\end{equation}

We can estimate the fifth term in the right hand side of (3.20) as follows

\begin{equation}
\ell_2(w, e_h) = \left| \sum_{T \in \mathcal{T}_h} \langle \nabla (\Delta w - Q_0 \Delta w) \cdot n, e_0 - e_b \rangle_{\partial T} \right|
\end{equation}

\begin{align*}
\leq & \left( \sum_{T \in \mathcal{T}_h} h_T \| \nabla (\Delta w - Q_0 \Delta w) \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \| e_0 - e_b \|_{\partial T}^2 \right)^{1/2} \\
\leq & Ch \| w \|_4 \| e_h \|.
\end{align*}

and for the last term,

\begin{equation}
\sum_{T \in \mathcal{T}_h} \langle \Delta w - Q_0 \Delta w, (\nabla e_0 - e_g) \cdot n \rangle_{\partial T}
\end{equation}

\begin{align*}
\leq & \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \| \Delta w - Q_0 \Delta w \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T \| \nabla e_0 - e_g \|_{\partial T}^2 \right)^{1/2} \\
\leq & Ch \| w \|_4 \| e_h \|.
\end{align*}

Substituting the estimates (3.21)–(3.26) into (3.20) and using (3.5) yield

\begin{equation}
\| e_0 \|^2 \leq Ch^2 \| u \|_4 \| w \|_4.
\end{equation}

Using the regularity estimate (3.17) we arrive at

\begin{equation}
\| e_0 \| \leq Ch^2 \| u \|_4.
\end{equation}

Using (3.19), the estimate above and the definition of $Q_0$, we prove the theorem. \hfill \Box

\section{Numerical results}

In this section, we shall perform the Weak Galerkin Algorithm 1 with $P_1$ element on (A) rectangular meshes, (B) uniform triangular meshes, (C) pentagonal meshes (hybrid with squares), (D) hexagonal meshes, and (E) criss-cross triangular meshes, cf. Fig. 3.1. Denote the mesh size (in $x$ or $y$ direction) as $h$. 
Table 4.1
Example 1: Error profiles and convergence rates.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u_0 - u|_0$ rate</th>
<th>$|\nabla(u_0 - u)|_0$ rate</th>
<th>$|u_0 - u|$ rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangular meshes (Fig. 3.1A)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>0.2586E-02 1.90</td>
<td>0.2451E-02 1.62</td>
<td>0.7791E-01 0.93</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>0.6906E-02 1.89</td>
<td>0.1058E-02 1.21</td>
<td>0.4035E-01 0.95</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>0.1809E-03 1.90</td>
<td>0.4092E-03 1.37</td>
<td>0.2070E-01 0.96</td>
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<tr>
<td>$2^{-7}$</td>
<td>0.4828E-04 1.95</td>
<td>0.1300E-03 1.65</td>
<td>0.1047E-01 0.98</td>
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<tr>
<td>Uniform triangular meshes (Fig. 3.1B)</td>
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<td></td>
<td></td>
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<tr>
<td>$2^{-4}$</td>
<td>0.1667E-02 1.88</td>
<td>0.2242E-02 1.44</td>
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<td>$2^{-5}$</td>
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<td>0.9270E-03 1.27</td>
<td>0.3244E-01 0.95</td>
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<tr>
<td>$2^{-6}$</td>
<td>0.1172E-03 1.94</td>
<td>0.3627E-03 1.35</td>
<td>0.1650E-01 0.98</td>
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<tr>
<td>$2^{-7}$</td>
<td>0.2974E-04 1.98</td>
<td>0.1496E-03 1.28</td>
<td>0.8302E-02 0.99</td>
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<td>0.9517E-02 0.22</td>
<td>0.2015E+00 0.64</td>
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<tr>
<td>$2^{-4}$</td>
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<td>0.7519E-02 0.34</td>
<td>0.1066E+00 0.92</td>
</tr>
<tr>
<td>$2^{-5}$</td>
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<td>0.4823E-03 0.64</td>
<td>0.5518E-01 0.95</td>
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<td>$2^{-6}$</td>
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<td>0.2866E-02 0.75</td>
<td>0.2845E-01 0.96</td>
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<tr>
<td>Hexagonal meshes (Fig. 3.1D)</td>
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<td>0.7526E-02 1.78</td>
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<td>$2^{-4}$</td>
<td>0.2028E-02 1.89</td>
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<td>0.1466E-02 0.90</td>
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<tr>
<td>Cross-cross triangular meshes (Fig. 3.1E)</td>
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<td>0.2802E-01 0.95</td>
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<tr>
<td>$2^{-6}$</td>
<td>0.8554E-04 1.96</td>
<td>0.3529E-03 1.22</td>
<td>0.1420E-01 0.98</td>
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Table 4.2
Example 2: Error profiles and convergence rates.

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<tr>
<th>$h$</th>
<th>$|u_0 - u|_0$ rate</th>
<th>$|\nabla(u_0 - u)|_0$ rate</th>
<th>$|u_0 - u|$ rate</th>
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</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Rectangular meshes (Fig. 3.1A)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>0.1568E-02 2.00</td>
<td>0.7540E-02 1.11</td>
<td>0.6132E-01 0.97</td>
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<tr>
<td>$2^{-5}$</td>
<td>0.3897E-03 2.01</td>
<td>0.3618E-02 1.06</td>
<td>0.3093E-01 0.99</td>
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<tr>
<td>$2^{-6}$</td>
<td>0.9695E-04 2.01</td>
<td>0.1773E-02 1.03</td>
<td>0.1553E-01 0.99</td>
</tr>
<tr>
<td>$2^{-7}$</td>
<td>0.2416E-04 2.00</td>
<td>0.8780E-03 1.01</td>
<td>0.7780E-02 1.00</td>
</tr>
<tr>
<td>Uniform triangular meshes (Fig. 3.1B)</td>
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</tr>
<tr>
<td>$2^{-4}$</td>
<td>0.9377E-03 2.02</td>
<td>0.4791E-02 1.04</td>
<td>0.5416E-01 0.97</td>
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<tr>
<td>$2^{-5}$</td>
<td>0.2320E-03 2.01</td>
<td>0.2339E-02 1.03</td>
<td>0.2740E-01 0.98</td>
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<td>Hexagonal meshes (Fig. 3.1D)</td>
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<td>0.1980E-02 0.98</td>
<td>0.1523E-01 1.00</td>
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<td>Cross-cross triangular meshes (Fig. 3.1E)</td>
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<td>0.2070E-02 1.00</td>
<td>0.1504E-01 0.99</td>
</tr>
</tbody>
</table>
4.1. Example 1

We solve the biharmonic equation (1.1) on the unit square domain $\Omega = (0, 1)^2$. The exact solution is

$$u = x^2(1 - x)^2y^2(1 - y^2).$$  \hfill (4.1)

The errors and the orders of convergence, on 5 types of grids, are listed in Table 4.1. They numerically verify the theorem.

4.2. Example 2

We solve the biharmonic equation (1.1) on the unit square domain $\Omega = (0, 1)^2$. The exact solution is

$$u = \sin x \cos y,$$  \hfill (4.2)

which does not satisfy the homogeneous boundary conditions. But the numerical computation is standard, computing a difference of the true solution and an arbitrary discrete function extending the boundary value.

The errors and the orders of convergence, on 5 types of grids, are listed in Table 4.2. We have one order triple-bar norm convergence and two orders $L^2$-norm convergence, verifying the theory.

References