Polynomial Preserving Recovery for Quadratic Elements on Anisotropic Meshes

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Polynomial preserving gradient recovery technique under anisotropic meshes is further studied for quadratic elements. The analysis is performed for highly anisotropic meshes where the aspect ratios of element sides are unbounded. When the mesh is adapted to the solution that has significant changes in one direction but very little, if any, in another direction, the recovered gradient can be superconvergent. The results further explain why recovery type error estimator is robust even under nonstandard and highly distorted meshes. © 2011 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 000: 000–000, 2011

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I. INTRODUCTION

Finite element recovery techniques are postprocessing methods that reconstruct numerical approximations from finite element solutions to achieve better results. Recent years witness a revitalization of this field, especially those years when Zienkiewicz-Zhu introduced their estimator based on superconvergence patch recovery (SPR), where they applied least squares fitting over a set of elements surrounding the vertex to smooth the stress (gradient) computed from finite element method [1, 2]. Later, another gradient recovery method, called polynomial preserving recovery (PPR) was proposed [3, 4]. Both theoretical analysis and numerical tests reveal that PPR has better or at least the same properties as SPR [5]. In this article, we study PPR for quadratic elements under anisotropic meshes. This is the further extension of a recent study for linear element [6]. Nevertheless, this extension is by no means straightforward. A newly developed tool by Huang–Xu [7] has to be adopted in order to carry on the needed analysis.
In this article, we consider superconvergence for quadratic elements. We know that the optimal convergence rate of gradient approximation by quadratic element is \(O(h^2)\). Our theoretical results show the convergence rate can reach \(O(h^3)\) for mildly structured meshes as well as anisotropic meshes with high aspect ratio in 2-D.

As for references regarding a posteriori error estimates and superconvergence related to this article, the reader is referred to [1, 2, 8–19].

II. PRELIMINARY KNOWLEDGE

A. Model Problem

On a bounded polytopic domain \(\Omega \subset \mathbb{R}^n\), we consider the boundary value problem: To find \(u \in H^1(\Omega)\) that satisfies some well-posed boundary conditions and for linear continuous functional \(f\) over \(H^1\)

\[
B(u, v) = f(v), \quad \forall v \in H^1(\Omega),
\]

with bilinear form defined by

\[
B(u, v) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{n} \int_{\Omega} b_i(x) u \frac{\partial v}{\partial x_i} dx + \int_{\Omega} cu v dx \nonumber
\]

\[
= \int_{\Omega} \nabla u \cdot A(x) \nabla v dx + (u, b \cdot \nabla v) + (cu, v).
\]

We assume the usual strong elliptic condition on \(A = A(x)\) and sufficient regularity on all input data, which will be specified later when needed.

Let \(T_h\) be a simplicial partition of \(\Omega\). The quadratic finite element space is then defined by

\[
S_h = \{ v : v \in H^1(\Omega), v|_\tau \in P_2, \forall \tau \in T_h \},
\]

where \(P_k\) is the set of polynomials of degree no more than \(k\). Then the finite element approximation \(u_h \in S_h\) satisfies

\[
B(u_h, v) = f(v), \quad \forall v \in S_h.
\]

B. Simplification

Let \(A_0\) be a piecewise constant function such that each element \(\tau \in T_h\)

\[
A_0|_\tau = \frac{1}{|\tau|} \int_\tau A(x) dx.
\]

Now we define

\[
a(u, v) = \int_{\Omega} \nabla u \cdot A(x) \nabla v dx, \quad a^\tau(u, v) = \int_{\tau} \nabla u \cdot A_0 \nabla v dx, \quad e_h = u - h_h,
\]

Assume that \(a_{ij} \in C^{0,\alpha}(\Omega)\), it is easy to show that

\[
|a(e_h, v) - \sum_{\tau} a^\tau(e_h, v)| \leq Ch^\alpha |e_h|_{1,\Omega} |v|_{1,\Omega}, \quad \alpha > 0.
\]
Therefore, we shift our analysis to $a^r(e_h, v)$. Since $A(x)$ is symmetric positive definite, so is $A_0|_{\tau}$. Then there exists an orthogonal matrix $Q_\tau$ such that $A_0|_{\tau} = Q_\tau^T D_\tau Q_\tau$ with $D_\tau = \text{diag}(d_1^\tau, \ldots, d_n^\tau)$. By changing of variable $x = Q_\tau z$, we have

$$a^r(e_h, v) = \int_{\tau} \nabla z e_h \cdot D_\tau \nabla z \det Q_\tau dz.$$  

Note that $\nabla z = Q_\tau \nabla x$, $\det Q_\tau = \pm 1$, and $\tau$ is obtained by rotating $\tau$. Therefore, without loss of generality, we may concentrate on the second-order form

$$\int_{\tau} \nabla e_h \cdot D_\tau \nabla v dx = \int_{\tau} \sum_{i=1}^n d_i^\tau \frac{\partial e_h}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \quad d_i^\tau > 0.$$  

and estimate the bilinear form

$$B(e_h, v) = \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \nabla e_h \cdot D_\tau \nabla v dx + \int_{\Omega} e_h (b \cdot \nabla v + cv) dx.$$  

Now, let’s focus on domain variation. Following [6], we assume that $\mathcal{T}_h$ can be separated into two parts

$$\mathcal{T}_h = \mathcal{T}_{0,h} \cup \mathcal{T}_{1,h}, \quad \bigcup_{\tau \in \mathcal{T}_{1,h}} \bar{\tau} = \bar{\Omega}_{1,h}, \quad \bar{\Omega} = \bar{\Omega}_{0,h} \cup \bar{\Omega}_{1,h},$$  

such that we have the following $\epsilon$-$\sigma$ condition:

1. Any two triangles that share a common edge in $\mathcal{T}_{0,h}$ form a convex quadrilateral which is an $\epsilon$-perturbation from a parallelogram.
2. $\Omega_{1,h}$ has a small measure: $|\Omega_{1,h}| = O(h^\sigma)(\sigma > 0)$.

Moreover, in the $y$-coordinate system, the image of $\Omega_{0,h}$ under the triangulation $\mathcal{T}_{0,h}$ is formed by triangles, where each pair of adjacent elements forms a parallelogram. Under this $y$-coordinate system, we define a “broken norm”

$$\|u\|_{k,\bar{\Omega}_y}^2 = \sum_{\tau \in \mathcal{T}_{0,h}} \|u\|_{k,\tau_y}^2.$$  

By a similar argument as in [6], we have the following theorem.

**Theorem 2.1.** Let $u \in H^4(\Omega) \cap W^2_\infty(\Omega)$ and $u_I \in S_h$ be the solution of the model problem and its quadratic finite element interpolant, respectively. Then

$$\left| \sum_{\tau \in \mathcal{T}_{0,h}} \int_{\tau} \nabla e_I \cdot D_\tau \nabla v dx \right| \lesssim (h^{5/2}(\|u\|_{4,\Omega_{0,h}} + \epsilon) + h^2 \epsilon(\|u\|_{3,\Omega_{0,h}} + \epsilon))|v|_{1,\Omega}. \quad (2.1)$$  

**Proof.** Let $e_I = u - u_I$, then, we have

$$\int_{\tau} \nabla e_I \cdot D_\tau \nabla v dx = I_0(r) + \epsilon I_1(r) + \epsilon^2 I_2(r). \quad (2.2)$$
Here,

\[ I_0(\tau) = \sum_{i=1}^{n} d_i \int_{\tau_i} \partial e_l \partial v \, dy = \int_{\tau} \nabla e_l \cdot D \nabla v \, dy; \]

\[ I_1(\tau) = \sum_{i=1}^{n} \int_{\tau_i} d_i \left( -\frac{\partial e_l}{\partial y_i} \frac{\partial v}{\partial y} \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial e_l}{\partial y_i} \right) dy; \]

\[ I_2(\tau) = \sum_{i=1}^{n} d_i \int_{\tau_i} \left( \sum_{k=1}^{n} \frac{\partial e_l}{\partial y_k} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \right) dy; \]

\( \epsilon \) is involved in the conformal transformation of domain variation

\[ y = x + \epsilon \eta(x), \]

where \( \eta \) is a piecewise linear function such that \( \nabla \eta \) is bounded uniformly in \( \epsilon \) and \( h \). Then, by the standard finite element superconvergence analysis, we are able to derive

\[ \left| \sum_{\tau \in T_{0,h}} I_0(\tau) \right| \lesssim h^{5/2} \| u \|_{4,\Omega} \| v \|_{1,\Omega} \lesssim h^{5/2} (\| u \|_{4,\Omega_{0,h}} + \epsilon) \| v \|_{1,\Omega}, \quad (2.3) \]

due to cancellations between parallel sides of adjacent triangles. For the term \( I_1(\tau) \), we have

\[ \left| \sum_{\tau \in T_{0,h}} I_1(\tau) \right| \lesssim h^2 \| u \|_{3,\Omega} \| v \|_{1,\Omega} \lesssim h^2 (\| u \|_{3,\Omega_{0,h}} + \epsilon) \| v \|_{1,\Omega}. \quad (2.4) \]

Combining (2.3)–(2.4), we obtain

\[ \left| \sum_{\tau \in T_{0,h}} \int_{\tau} \nabla e_l \cdot D \nabla v \, dx \right| \lesssim (h^{5/2} (\| u \|_{4,\Omega_{0,h}} + \epsilon) + h^2 \epsilon (\| u \|_{3,\Omega_{0,h}} + \epsilon)) \| v \|_{1,\Omega}. \quad (2.5) \]

**Polynomial Preserving Gradient Recovery.** Following [3, 4], we introduce a gradient recovery operator \( G_h : S_h \rightarrow S_h \times S_h \), which has the two properties below:

1. Polynomial preserving:
   \[ G_h(p_i) = \nabla p, \quad \forall p \in P_3. \]
   Consequently, it owns the approximation property
   \[ \| \nabla u - G_h u_1 \| \lesssim h^3 \| u \|_{4,\Omega}, \quad \forall u \in H^4(\Omega). \]

2. Boundedness: When there are no two adjacent angles on an element patch adding up to exceed \( \pi \), we have
   \[ \| G_h v \| \lesssim \| v \|_{1,\Omega}, \quad \forall v \in S_h. \]
Theorem 2.2. Let \( u \in H^4(\Omega) \cap W^3_\infty(\Omega) \) and \( u_h \in S_h \) be the solution of the model problem and its quadratic finite element approximation respectively. Assume (a) the \( \epsilon-\sigma \) mesh condition, (b) the maximum angle condition, and (c) the discrete inf-sup condition. Then the polynomial preserving recovery operator \( G_h \) leads to superconvergence in the sense that:

\[
\| \nabla u - G_h u_h \| \lesssim h^3 \| u \|_{4, \Omega} + h^{5/2} (\| u \|_{4, \Omega_0, h} + \epsilon) + h^2 \epsilon (\| u \|_{3, \Omega_0, h} + \epsilon) + h^{2+\sigma/2} \| u \|_{3, \infty, \Omega}.
\]

Proof. By the triangle inequality and the polynomial preserving property,

\[
\| \nabla u - G_h u_h \| \leq \| \nabla u - G_h u_I \| + \| G_h (u_I - u_h) \| \lesssim h^3 \| u \|_{4, \Omega} + \| u_h - u_I \|_{1, \Omega}.
\]

The analysis for the second term on the right-hand side under conditions (a) and (b) is proceeded as the following:

\[
\sum_{\tau \in T_{1,h}} \int_{\tau} \nabla e_1 \cdot D_\tau \nabla v dx \lesssim h^3 \| u \|_{3, \infty, \Omega} \sum_{\tau \in T_{1,h}} \int_{\tau} |\nabla v| dx \lesssim h^{2+\sigma/2} \| u \|_{3, \infty, \Omega} \| \nabla v \|_{0, \Omega},
\]

and (by the standard approximation theory)

\[
\int_{\Omega} e_1 (b \cdot \nabla v + cv) dx \lesssim h^3 \| u \|_{3, \Omega} \| v \|_{1, \Omega}.
\]

Based on (2.5), (2.7), and (2.8), we derive

\[
|B(e_1, v)| \lesssim (h^{5/2} (\| u \|_{4, \Omega_0, h} + \epsilon) + h^2 \epsilon (\| u \|_{3, \Omega} + \epsilon) + h^{2+\sigma/2} \| u \|_{3, \infty, \Omega}) |v|_{1, \Omega}.
\]

Using the discrete inf-sup condition and the strong elliptic assumption, we then have

\[
\| u_I - u_h \|_{1, \Omega} \lesssim \sup_{v \in S_h} \frac{B(u_h - u_I, v)}{\| v \|_{1, \Omega}} = \sup_{v \in S_h} \frac{B(e_1, v)}{\| v \|_{1, \Omega}} \\
\lesssim h^{5/2} (\| u \|_{4, \Omega_0, h} + \epsilon) + h^2 \epsilon (\| u \|_{3, \Omega} + \epsilon) + h^{2+\sigma/2} \| u \|_{3, \infty, \Omega}.
\]

The conclusion follows by substituting (2.10) into (2.6).

III. ERROR ESTIMATES UNDER ANISOTROPIC MESHES

We analyze errors under triangulation of the regular and Union-Jack patterns. We consider the worst case for the regular pattern where the maximum angle condition is violated at the extremal case \( \theta \to 0 \), where a patch is compressed at a fixed ratio, i.e., \( \theta_2 = k_2 \theta, \theta_3 = k_3 \theta, \) and \( k_2, k_3 > 0 \).

To simplify the matter, we analyze the case \( b = 0 \) and \( c = 0 \). We need the following integral identity [7] for \( v_Q \in P_2(\tau) \),

\[
\int_{\tau} \nabla (u - u_I) \cdot \nabla v_Q = \sum_{k=1}^{3} \sum_{s=0}^{3} \left( a_s^I(\tau) \int_{\tau} + b_s^I(\tau) \int_{l_k} \right) \frac{\partial^3 u}{\partial n_{l_k} \partial t_s^3} \frac{\partial^2 v_Q}{\partial t_s^3} \\
+ O(h^{4-j}) \| u \|_{4, p, \tau} |v_Q|_{2-j, q, \tau}, \quad j = 0, 1.
\]
where \( k \) is modulo 3, angle \( \theta_k \) is opposite of side \( l_k \) (with length \( \ell_k \)), \( t_k (n_k) \) is the counter-clockwise unit tangential (outer-normal) vector on side \( l_k \). Denote

\[
M_k = \frac{\sin 2\theta_k}{\sin 2\theta_1 + \sin 2\theta_2 + \sin 2\theta_3}.
\]

We can express

\[
d_k^0(\tau) = M_k \left( \frac{l_{k-1}^2 l_k}{180} \cos^2 \theta_{k+1} + \frac{l_1 l_2 l_3}{120} \cos \theta_{k-1} \cos \theta_{k+1} + \frac{l_{k+1}^2 l_{k-1}}{180} \cos \theta_{k-1} \cos \theta_{k+1} \right),
\]

\[
d_k^1(\tau) = M_k \left( -\frac{l_{k-1}^2 l_k}{90} \sin \theta_{k+1} \cos \theta_{k+1} + \frac{l_1 l_2 l_3}{120} \cos \theta_{k-1} \sin \theta_{k+1} \right.
\[
- \frac{l_{k+1}^2 l_{k-1}}{90} \cos \theta_{k-1} \sin \theta_{k+1} \cos \theta_{k+1} + \frac{l_1 l_2 l_3}{120} \cos \theta_{k+1} \sin \theta_{k-1} \left.
\right)
\]

\[
d_k^2(\tau) = M_k \left( \frac{l_{k-1}^2 l_k}{180} \sin^2 \theta_{k+1} - \frac{l_1 l_2 l_3}{120} \sin \theta_{k-1} \sin \theta_{k+1} + \frac{l_{k+1}^2 l_{k-1}}{180} \sin \theta_{k-1} \sin \theta_{k+1} \right)
\]

\[
d_k^3(\tau) = M_k \frac{l_{k+1}^2 l_{k-1}}{180} \sin^2 \theta_{k-1} \sin \theta_{k+1},
\]

and

\[
b_k^0(\tau) = \frac{1}{1440} \left( \frac{l_k^4 (l_k^2 - l_{k-1}^2)}{2|\tau|} + \frac{l_{k-1}^4 \cos^3 \theta_{k+1}}{\sin \theta_{k+1}} - \frac{l_{k+1}^4 \cos^3 \theta_{k-1}}{\sin \theta_{k-1}} \right),
\]

\[
b_k^1(\tau) = \frac{1}{1440} \left( -3l_{k-1}^4 \cos^2 \theta_{k+1} - 3l_{k+1}^4 \cos^2 \theta_{k-1} \right),
\]

\[
b_k^2(\tau) = \frac{1}{1440} \left( 3l_{k+1}^4 \cos \theta_{k+1} \sin \theta_{k+1} - 3l_{k+1}^4 \cos \theta_{k-1} \sin \theta_{k-1} \right),
\]

\[
b_k^3(\tau) = \frac{1}{1440} \left( -l_{k+1}^4 \sin^2 \theta_{k-1} - l_{k-1}^4 \sin^2 \theta_{k+1} \right).
\]

In Fig. 1, we let \( \theta_2 = k_2 \), \( \theta_3 = k_3 \). It is straightforward to verify

\[
\frac{\partial^3 u}{\partial t^3} = \frac{\partial^3 u}{\partial x^3},
\]

\[
\frac{\partial^3 u}{\partial t^2} = -\cos^3 k_3 \frac{\partial^3 u}{\partial x^3} + (3 \cos^2 k_3 \theta \sin k_3 \theta) \frac{\partial^3 u}{\partial x^2 \partial y}
\]

\[
- (3 \cos k_3 \theta \sin^2 k_3 \theta) \frac{\partial^3 u}{\partial x \partial y^2} + \sin^3 k_3 \theta \frac{\partial^3 u}{\partial y^3}.
\]
But independent of $h$. Here and hereafter, if $\text{Lemma 3.1.}$ in above have the following properties:

$$(1) \quad a_k^0, a_k^1, b_k^0, \quad k = 1, 2, 3, \quad \text{are balanced, then the coefficients } a_k^1(\tau) \text{ and } b_k^1(\tau) \text{ given in above have the following properties:}$$

$$a_k^1(\tau) = O(h^3), \quad b_k^1(\tau) = O(h^4);$$

Note that $\sin \theta = \sin(\pi - \theta_2 - \theta_3) = \sin(k_2 + k_3)\theta$. Let $\ell_1 = h, \ell_2 \sim h, \ell_3 \sim h$. Therefore, as $\theta \to 0$, for $k = 1, 2, 3$, we have

$$1 : a_k^0 \sim h^3\theta^{-2}, \quad 2 : a_k^1 \sim h^3\theta^{-1}, \quad 3 : a_k^2 \sim h^3, \quad 4 : a_k^3 \sim h^3\theta, \quad 5 : b_k^0 \sim h^4\theta^{-1}, \quad 6 : b_k^1 \sim h^4, \quad 7 : b_k^2 \sim h^4\theta, \quad 8 : b_k^3 \sim h^4\theta^2. \quad (3.1)$$

To balance $a_k^0, a_k^1$ and $b_k^0$, we need following conditions:

$$\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \tau} = C(u, \tau)\theta^2, \quad \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \tau} = C(u, \partial \tau)\theta, \quad \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0, \tau} = C(u, \tau)\theta. \quad (3.2)$$

Here and hereafter, $C(u, \tau)(C(u, \partial \tau))$ means a generic constant that depends on $u$ and $\tau(\partial \tau)$, but independent of $h$ and $\theta$. 

**Lemma 3.1.** If $a_k^0, a_k^1, b_k^0, \quad k = 1, 2, 3$, are balanced, then the coefficients $a_k^1(\tau)$ and $b_k^1(\tau)$ given in above have the following properties:

$$(1) \quad a_k^1(\tau) = O(h^3), \quad b_k^1(\tau) = O(h^4);$$
(2) Assume that $\tau$ and $\tau'$ are two adjacent equivalent triangles. Then

$$a^*_k(\tau) = a^*_k(\tau'), \quad b^*_k(\tau) = b^*_k(\tau');$$

(3) If $\tau$ and $\tau'$ form a pair of $O(h^2)$ approximate equivalent triangles (namely the lengths of any two opposite edges differ only by $O(h^2)$), then

$$a^*_k(\tau) - a^*_k(\tau') = O(h^4), \quad b^*_k(\tau) - b^*_k(\tau') = O(h^5).$$

Also, by direct computation, one easily get

$$\frac{\partial^2 v_Q}{\partial t^2} \sim \frac{1}{h^2}, \quad k = 1, 2, 3. \quad (3.3)$$

$$\frac{\partial v_Q}{\partial t_k} \sim \frac{1}{h}, \quad k = 1, 2, 3. \quad (3.4)$$

Thus,

$$\left| \frac{\partial^2 v_Q}{\partial t_k^2} \right|_{L^2(\tau)} \lesssim h^{-1} \left| \frac{\partial v_Q}{\partial t_k} \right|_{L^2(\tau)}, \quad k = 1, 2, 3. \quad (3.5)$$

Hence, by Lemma 3.1, Hölder’s inequality and (3.3)–(3.5), we have

$$\left( a^0_1(\tau) \int_{\tau} - a^0_1(\tau') \int_{\tau'} \right) \frac{\partial^3 u}{\partial t_1^3} \frac{\partial^2 v_Q}{\partial t_1^2} \lesssim h^3 \theta^{-2} C(u, \tau) \theta^2 |v_Q|_{1, \tau} \leq h^3 C(u, \tau) \|v_Q\|_{1, \tau}.$$
However, we can not obtain a similar estimate for boundary terms under such conditions as above because

\[
\left| \frac{\partial v_Q}{\partial t_k} \right|_{L^2(\gamma)} \lesssim h^{-1/2} \theta^{-1/2} \left| \frac{\partial v_Q}{\partial t_k} \right|_{L^2(\tau)}. \tag{3.6}
\]

To get rid of \( \theta^{-1/2} \), condition (3.2) is refined to

\[
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \tau} = C(u, \tau) \theta^2, \quad \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0, \tau} = C(u, \tau) \theta,
\]

\[
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \sigma} = C(u, \sigma) \theta^{3/2}, \quad \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0, \sigma} = C(u, \sigma) \theta^{1/2}. \tag{3.7}
\]

Thus, assume condition (3.7), and (3.3)–(3.5), we obtain:

\[
\left( b_1^0(\tau) - b_1^0(\tau') \int_{l_1} \right) \frac{\partial^3 u}{\partial t_1^3} \frac{\partial^2 v_Q}{\partial t_1^2} \lesssim h^5 \theta^{-1} C(u, l_1) \theta^{3/2} \left| \frac{\partial^2 v_Q}{\partial t_1^2} \right|_{L^2(\tau)} \tag{By (3.1)}
\]

\[
\lesssim h^{9/2} C(u, l_1) \left| \frac{\partial^2 v_Q}{\partial t_1^2} \right|_{L^2(\tau)} \tag{By (3.6)}
\]

\[
\lesssim h^{7/2} C(u, l_1) v_Q |_{1, \tau} \tag{By (3.5)}
\]

\[
\lesssim h^{7/2} C(u, l_1) \| v_Q \|_{1, \tau}. \tag{By (3.5)}
\]

By following the same method, we can obtain the estimate for the rest of cases. For the sake of demonstration, we provide the proof of one more case here.

\[
\left( b_2^0(\tau) - b_2^0(\tau') \int_{l_2} \right) \frac{\partial^3 u}{\partial t_2^3} \frac{\partial^2 v_Q}{\partial t_2^2} \lesssim h^5 C(u, l_2) \theta^{1/2} \left| \frac{\partial^2 v_Q}{\partial t_2^2} \right|_{L^2(\tau)} \tag{By (3.1)}
\]

\[
\lesssim h^{9/2} C(u, l_2) \left| \frac{\partial^2 v_Q}{\partial t_2^2} \right|_{L^2(\tau)} \tag{By (3.6)}
\]

\[
\lesssim h^{7/2} C(u, l_2) v_Q |_{1, \tau} \tag{By (3.5)}
\]

\[
\lesssim h^{7/2} C(u, l_2) \| v_Q \|_{1, \tau}. \tag{By (3.5)}
\]

Hence, for any \( k = 1, 2, 3 \) and \( s = 0, 1, 2 \), we have

\[
\left( b_k^s(\tau) - b_k^s(\tau') \int_{l_k} \right) \frac{\partial^3 u}{\partial n_k^s \partial t_k^{1-s}} \frac{\partial^2 v_Q}{\partial t_k^2} \lesssim h^{7/2} C(u, \gamma) \| v_Q \|_{1, \tau},
\]

where \( \gamma \) is the common edge of \( \tau \) and \( \tau' \). In addition, due to cancellations between parallel sides of adjacent triangles, we have the following identity [7]:

\[
(\nabla (u - u_I), \nabla v_h) = \sum_{s=0}^{3} \sum_{e=\Gamma(\tau')} \left( a_e^s(\tau) \int_{\tau'} - a_e^s(\tau') \int_{\tau'} \right) + \left[ b_e^s(\tau) - b_e^s(\tau') \right] \int_{\tau'} \frac{\partial^3 u}{\partial n_k^s \partial t_k^{1-s}} \frac{\partial^2 v_Q}{\partial t_k^2} + O(h^{4-s}) |u|_{4,p, \Omega} v_h |_{2-s, q, \Omega}.
\]

Following the above identity, we derive the following theorem.
Theorem 3.1. Let \( u \in H^4(\Omega_a) \) and let \( \Omega_a \subset \Omega \) contain anisotropic uniform triangles of type in Fig. 1 with \( \theta \in (0, \pi/4] \) (\( \theta \in (\pi/4, \pi/2) \) can be treated similarly by shifting the focus directions). Assume that
\[
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \tau} = C(u, \tau) \theta^2, \quad \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0, \tau} = C(u, \tau) \theta,
\]
\[
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \partial \tau} = C(u, \partial \tau) \theta^{3/2}, \quad \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0, \partial \tau} = C(u, \partial \tau) \theta^{1/2}.
\]

Then we have
\[
\left| \sum_{\tau \subset \Omega_a} \int_{\tau} \nabla e_I \cdot D_v \nabla v_h \right| \lesssim h^3 |v_h|_{1, \Omega_a}, \quad \forall v_h \in S_h^{0}(\Omega_a).
\]

Remark 1. From Theorem 3.1, we see that if \( u \) has very little activity in the \( x \)-direction, the degenerating limit \( \theta = 0 \) in a sample triangle \( \tau \) can be allowed, which is equivalent to allow the maximum angle in \( \tau \) be as close as possible to \( \pi \). The rate of convergence is maintained at \( O(h^3) \).

Remark 2. In this case, two adjacent triangles form a parallelogram or a quasi-parallelogram, so cancellation instead of doubling between parallel sides occurs.

Remark 3. In the result, the constant depends on \( u \) and \( \Omega_a \), but it is independent of \( h \) and \( \theta \).

Next, let us consider the model problem under Union-Jack Patch, see Fig. 2.
In this case, we can not analyze the problem in a similar fashion as we did in the previous case since some two adjacent triangles, say \( \tau_1 \) and \( \tau_5 \) does not form a quasi-parallelogram. The integral on the side shared by these two triangles doubles instead of canceling. However, if we group triangles \( \tau_1 \) and \( \tau_5 \), \( \tau_2 \) and \( \tau_6 \), \( \tau_3 \) and \( \tau_7 \), and \( \tau_4 \) and \( \tau_8 \), cancellations also happens under certain conditions. Here we only focus on \( \tau_1 \) and \( \tau_5 \) and we allow them to form a pair of \( O(h^2) \) equivalent triangles, but we restrict \( \theta_2 = \frac{\pi}{2} \). For other pairs, we can analyze them in exactly the same way (Fig. 3).

From the graph above, we know that \( \theta_1, \theta_2, \) and \( \theta_3 \) of these two triangles are almost the same. Moreover, both of these pairs are in count-clockwise order. Hence,
\[
a^s_k(\tau_1) - a^s_k(\tau_5) = O(h^4), \quad b^s_k(\tau_1) - b^s_k(\tau_5) = O(h^5), \quad k = 1, 2, 3; s = 0, 1, 2, 3.
\]

Also, both the unit tangent and out-normal vectors on the corresponding sides are opposite.

We also assume that the patch is compressed at a fixed ratio as in the case of regular mesh, i.e., \( \theta_3 = k \theta \). Hence, \( l_1 = h, l_2 = O(h), l_3 = O(h) \theta, \) as \( \theta \to 0 \). By simple computation, we obtain

1. \( a_0^0 \sim h^3, \quad a_0^1 = 0, \quad a_0^2 \sim h^3 \theta^3, \)
2. \( a_1^0 \sim h^3 \theta, \quad a_1^1 = 0, \quad a_1^2 \sim h^3 \theta^2, \)
3. \( a_2^0 \sim h^3 \theta^2, \quad a_2^1 = 0, \quad a_2^2 \sim h^3, \)
4. \( b_0^0 \sim h^4 \theta^{-1}, \quad b_0^1 \sim h^4 \theta^{-1}, \quad b_0^2 \sim h^4 \theta^3, \)
5. \( b_1^0 \sim h^4, \quad b_1^1 \sim h^4, \quad b_1^2 \sim h^4 \theta^2, \)
6. \( b_2^0 \sim h^4 \theta, \quad b_2^1 \sim h^4 \theta, \quad b_2^2 \sim h^4 \theta, \)
7. \( b_3^0 \sim h^4 \theta^2, \quad b_3^1 \sim h^4 \theta^2, \quad b_3^2 \sim h^4. \)

On the other hand,

\[
\frac{\partial^3 u}{\partial t^3} = \frac{\partial^3 u}{\partial x^3},
\]
\[
\frac{\partial^3 u}{\partial n^3} = -\frac{\partial^3 u}{\partial x^3},
\]
\[
\frac{\partial^3 u}{\partial t^2} = -\cos^3 k \theta \frac{\partial^3 u}{\partial x^3} + (3 \cos^2 k \theta \sin k \theta) \frac{\partial^3 u}{\partial x^2 \partial y} - (3 \cos k \theta \sin^2 k \theta) \frac{\partial^3 u}{\partial x \partial y^2} + \sin^3 k \theta \frac{\partial^3 u}{\partial y^3},
\]
\[
\frac{\partial^3 u}{\partial n^2 t^2} = \cos^2 k \theta \sin k \theta \frac{\partial^3 u}{\partial x^3} + (\cos^3 k \theta - 2 \sin^2 k \theta \cos k \theta) \frac{\partial^3 u}{\partial x^2 \partial y}
+ (\sin^3 k \theta - 2 \cos^2 k \theta \sin k \theta) \frac{\partial^3 u}{\partial x \partial y^2} + \sin^3 k \theta \cos \theta \frac{\partial^3 u}{\partial y^3},
\]

It follows that if

\[
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{\tau} = C(u, \partial \tau) \theta,
\]

then \( b_1^0 \) and \( b_2^0 \) are balanced. From Appendix, we see that under the Union-Jack mesh,

\[
\left| \frac{\partial^2 v}{\partial t^2_k} \right|_{L^2(\tau)} \lesssim h^{-1}\theta^{-1} |v|_{1, \tau} \lesssim h^{-1}\theta^{-1} \|v\|_{1, \tau}, \quad k = 1, 2, 3;
\]

\[
\left| \frac{\partial^2 v}{\partial t^2_k} \right|_{L^2(\gamma')} \lesssim h^{-3/2}\theta^{-1} |v|_{1, \tau} \lesssim h^{-3/2}\theta^{-1} \|v\|_{1, \tau}, \quad k = 1, 2, 3. \tag{3.10}
\]

Thus, we require condition (3.9) to be refined to

\[
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \partial \tau} = C(u, \partial \tau)\theta, \quad \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \partial \tau} = C(u, \partial \tau)\theta, \quad \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0, \partial \tau} = C(u, \partial \tau)\theta. \tag{3.11}
\]

Hence, if \( u \in W^{4, r}(\Omega) \) and condition (3.11) holds, then we obtain

\[
\left( a_k^1(\tau_1) \int_{\tau_1} + a_k^3(\tau_3) \int_{\tau_3} \right) \frac{\partial^3 u}{\partial n_k^3 \partial t_k^{3-s}} \frac{\partial^2 v_Q}{\partial t_k^2} = a_k^3(\tau_1) \left\{ \int_{\tau_1} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^{3-s}} \frac{\partial^2 v_Q}{\partial t_k^2} + \int_{\tau_3} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^{3-s}} \frac{\partial^2 v_Q}{\partial t_k^2} \right\} + (a_k^3(\tau_3) - a_k^3(\tau_1)) \int_{\tau_3} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^{3-s}} \frac{\partial^2 v_Q}{\partial t_k^2} = I_1 + I_2.
\]

Since \( n_k \) and \( t_k \) of corresponding sides of \( \tau_1 \) and \( \tau_3 \) are opposite, we conclude that the sign of

\[
\int_{\tau_1} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^{3-s}} \frac{\partial^2 v_Q}{\partial t_k^2} \quad \text{and} \quad \int_{\tau_3} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^{3-s}} \frac{\partial^2 v_Q}{\partial t_k^2}
\]

are opposite.

Therefore, for \( k = 1 \) and \( s = 0 \), we obtain

\[
I_1 \lesssim h |a_1^0(\tau)| \cdot C(u, \tau_1 \cup \tau_3)\theta \cdot \left| \frac{\partial^2 v_Q}{\partial t_k^2} \right|_{0, \partial \tau_1 \cup \tau_3} \quad \text{(By (3.8) and (3.11))}
\]

\[
\lesssim h^4 C(u, \tau_1 \cup \tau_3)\theta \cdot \left| \frac{\partial^2 v_Q}{\partial t_k^2} \right|_{0, \partial \tau_1 \cup \tau_3} \quad \text{(By (3.10))}
\]

\[
I_2 \lesssim h^4 \cdot C(u, \tau_1 \cup \tau_3)\theta \cdot \left| \frac{\partial^2 v_Q}{\partial t_k^2} \right|_{0, \partial \tau_1 \cup \tau_3} \quad \text{(By (3.8) and (3.11))}
\]

\[
\lesssim h^4 C(u, \tau_1 \cup \tau_3)\theta \cdot \left| \frac{\partial^2 v_Q}{\partial t_k^2} \right|_{0, \partial \tau_1 \cup \tau_3} \quad \text{(By (3.10))}
\]

By following the same argument, we can obtain the same estimate for all other \( k \)’s and \( s \)’s. Hence,

\[
I_i \lesssim h^3 C(u, \tau_1 \cup \tau_3) \|v\|_{1, \tau_1 \cup \tau_3}, \quad k = 1, 2, 3, \quad s = 0, 1, 2, 3, \quad i = 1, 2.
\]
Summing up $I_1$ and $I_2$, we obtain,

\[
\left( a^*_1(\tau_1) \int_{\tau_1} + a^*_5(\tau_5) \int_{\tau_5} \right) \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} \lesssim h^3 C(u, \tau_1 \cup \tau_5) \| v \|_{1, \tau_1 \cup \tau_5}.
\]

Now, let’s consider terms involving integration on the boundary.

\[
\left( b^*_1(\tau_1) \int_{l_1} + b^*_5(\tau_5) \int_{l_5} \right) \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} = b^*_1(\tau_1) \left\{ \int_{l_1} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} + \int_{l_5} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} \right\} + \left( b^*_5(\tau_5) - b^*_1(\tau_1) \right) \int_{l_5} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} = L_1 + L_2.
\]

By the same analysis as $I_1$, we know that the sign of

\[
\int_{l_1} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} \quad \text{and} \quad \int_{l_5} \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2}
\]

are opposite. Hence, for $k = 1$ and $s = 0$,

\[
L_1 \lesssim h \left| b^0_1(\tau) \right| \cdot C(u, l_1 \cup l_5) \theta^2 \cdot \left| \frac{\partial^2 v_Q}{\partial t_1^2} \right|_{0, l_1 \cup l_5}, \quad \text{(By (3.8) and (3.11))}
\]

\[
\lesssim h^{7/2} C(u, l_1 \cup l_5) \left| \frac{\partial^2 v_Q}{\partial t_1^2} \right|_{0, \tau}, \quad \text{(By (3.10))}
\]

\[
\lesssim h^{7/2} C(u, l_1 \cup l_5) \| v_Q \|_{1, \tau}.
\]

\[
L_2 \lesssim h^5 \theta^{-1} \cdot C(u, l_1 \cup l_5) \theta^2 \cdot \left| \frac{\partial^2 v_Q}{\partial t_1^2} \right|_{0, l_1 \cup l_5}, \quad \text{(By (3.8) and (3.11))}
\]

\[
\lesssim h^{7/2} C(u, l_1 \cup l_5) \left| \frac{\partial^2 v_Q}{\partial t_1^2} \right|_{0, \tau}, \quad \text{(By (3.10))}
\]

\[
\lesssim h^{7/2} C(u, l_1 \cup l_5) \| v_Q \|_{1, \tau}.
\]

Thus,

\[
\left( b^0_1(\tau_1) \int_{l_1} + b^0_5(\tau_5) \int_{l_5} \right) \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} \lesssim h^{7/2} C(u, l_1 \cup l_5) \| v_Q \|_{1, \tau}.
\]

By the same argument as above, we can obtain

\[
\left( b^*_k(\tau_1) \int_{l_1} + b^*_k(\tau_5) \int_{l_5} \right) \frac{\partial^3 u}{\partial n_k^3 \partial t_k^3} \frac{\partial^2 v_Q}{\partial t_k^2} \lesssim h^{7/2} C(u, l_1 \cup l_5) \| v_Q \|_{1, \tau_1 \cup \tau_5} \quad k = 1, 2, 3, \ s = 0, 1, 2, 3.
\]

Again, we provide the proof of one more case for demonstration. For $k = 2, s = 1$,

$$L_1 \lesssim h \left| b_1^1(\tau) \right| \cdot C(u, l_1 \cup l_3) \theta \cdot \left| \frac{\partial^2 v_Q}{\partial t^2} \right|_{0, l_1 \cup l_3}, \text{ (By (3.8) and (3.11))}$$

$$\lesssim h^{7/2} C(u, l_1 \cup l_3) \left| \frac{\partial^2 v_Q}{\partial t^2} \right|_{0, \tau}, \text{ (By (3.10))}$$

$$L_2 \lesssim h^5 \cdot C(u, l_1 \cup l_3) \theta \cdot \left| \frac{\partial^2 v_Q}{\partial t^2} \right|_{0, l_1 \cup l_3}, \text{ (By (3.8) and (3.11))}$$

$$\lesssim h^{7/2} C(u, l_1 \cup l_3) \left| \frac{\partial^2 v_Q}{\partial t^2} \right|_{0, \tau}, \text{ (By (3.10))}$$

$$\lesssim h^{7/2} C(u, l_1 \cup l_3) \| v_Q \|_{1, \tau}.$$ 

Our result can be concluded as the following theorem:

**Theorem 3.2.** Let $u \in H^5(\Omega_a)$ and let $\Omega$ contain anisotropic uniform triangles of type in Figure 2 with $\theta \in (0, \pi/4]$ ($\theta \in (\pi/4, \pi/2)$ can be treated similarly by shifting the focus directions). Assume that

$$\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \tau} = C(u, \partial \tau) \theta, \quad \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0, \tau} = C(u, \tau) \theta, \quad \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0, \tau} = C(u, \partial \tau) \theta.$$ 

Then we have

$$\sum_{\tau \subset \Omega_a} \int_{\tau} \nabla e_{\tau} \cdot D_{\tau} \nabla v_h \lesssim h^3 \| v_h \|_{1, \Omega_a}, \quad \forall v_h \in S_h^0(\Omega_a).$$

By the similar argument as in Theorem 2.2, we have following theorems.

**Theorem 3.3.** Let $u \in H^4(\Omega) \cap W^3_{\infty}(\Omega)$ and $u_h \in S_h$ be the solution of the model problem and its quadratic finite element approximation, respectively. Assume (a) $\epsilon$-$\sigma$ mesh condition, (b) $\Omega_a \subset \Omega_{0, h}$ contains anisotropic uniform triangles of type in Fig. 1 with $\theta \in (0, \pi/4]$ and elements in $\Omega \setminus \Omega_a$ satisfy the maximum angle condition, and (c) the discrete inf-sup condition. Furthermore, assume that the condition in Theorem 3.1 is satisfied. Then the polynomial preserving gradient recovery operator $G_h$ has the following error bound:

$$\| \nabla u - G_h u_h \| \leq h^3 C_1(u, \Omega) + h^{5/2} (C_2(u, \Omega_{0, h}) + \epsilon) + h^2 \epsilon (C_3(u, \epsilon) + \epsilon) + h^{2+\sigma/2} C_4(u, \Omega).$$

**Theorem 3.4.** Let $u \in H^5(\Omega) \cap W^3_{\infty}(\Omega)$ and $u_h \in S_h$ be the solution of the model problem and its quadratic finite element approximation, respectively. Assume (a) $\epsilon$-$\sigma$ mesh condition, (b) $\Omega_a \subset \Omega_{0, h}$ contains anisotropic uniform triangles of type in Fig. 2 with $\theta \in (0, \pi/4]$ and elements in $\Omega \setminus \Omega_a$ satisfy the maximum angle condition, and (c) the discrete inf-sup condition. Furthermore, assume that the condition in Theorem 3.2 is satisfied. Then the polynomial preserving gradient recovery operator $G_h$ has the following error bound:

$$\| \nabla u - G_h u_h \| \leq h^3 C_1(u, \Omega) + h^{5/2} (C_2(u, \Omega_{0, h}) + \epsilon) + h^2 \epsilon (C_3(u, \epsilon) + \epsilon) + h^{2+\sigma/2} C_4(u, \Omega).$$

IV. NUMERICAL EXPERIMENT

We compute the Neumann boundary equation below on the unit square domain:

\[
\begin{cases}
-\Delta u + u = f, \\
\frac{\partial u}{\partial \mu} = g.
\end{cases}
\]  

(4.1)

Here, we choose proper functions \( f \) and \( g \) such that \( u = \sin(\pi x) \sin(10\pi y) \). The \( L_2 \) Error on the whole domain under regular and Union-Jack patterns are shown in Table I:

Corresponding graph is in Fig. 4.

APPENDIX: INVERSE INEQUALITY FOR UNION-JACK ANISOTROPIC MESHES

Let us focus on one \( \tau \) in Fig. A1 since we can obtain the results for regular mesh in a similar way. We observe that \( h = \theta H \) as \( \theta \to 0 \).

Theorem. Let \( \gamma \) be the boundary of \( \tau \), and \( v \in S_h \), then

\[
\left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\tau)} \lesssim \frac{1}{H\theta} \| v \|_{1,\tau}
\]

16 HUANG AND ZHANG

FIG. A1. One element of Union-Jack pattern.

and

$$\left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\tau)} \lesssim \frac{1}{H^{3/2} \theta} \| v \|_{1, \tau}$$

Proof. Without loss of generality, we assume that vertexes of \( \tau \) are \( A(0,0) \), \( B(H,0) \), and \( C(0,h) \). Then \( v \) is of the form

$$v = a \left( \frac{x}{H} \right)^2 + b \left( \frac{x}{H} \right) \left( \frac{y}{h} \right) + c \left( \frac{y}{h} \right)^2 + d \frac{x}{H} + e \frac{y}{h} + f,$$

where \( a, b, c, d, e, f \) are constants.

The results are obvious if \( v \) is a constant or a linear function, then the results follow from the classic inverse inequality. From now on, we assume that the coefficient of \( y^2 \) term is not 0, i.e., \( c \neq 0 \).

$$v_x = \frac{2ax}{H^2} + \frac{by}{Hh} + \frac{d}{H}, \quad v_y = \frac{2cy}{h^2} + \frac{bx}{Hh} + \frac{e}{h},$$

$$v_{xx} = \frac{2a}{H^2}, \quad v_{yy} = \frac{b}{h^2}, \quad v_{xy} = \frac{2c}{Hh},$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \cos^2 \theta \frac{\partial^2 v}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 v}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 v}{\partial y^2}.$$  

Therefore,

$$\left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\tau)} \sim H^{-2} \cdot \sqrt{hH} = \frac{h^{1/2}}{H^{3/2}},$$

$$\left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\tau)} \sim H^{-2} \cdot H^{1/2} = H^{-3/2},$$

$$\left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\tau)} \leq \left\| \frac{\partial^2 v}{\partial x^2} \right\|_{L^2(\tau)} + \left\| \frac{\partial^2 v}{\partial x \partial y} \right\|_{L^2(\tau)} + \left\| \frac{\partial^2 v}{\partial y^2} \right\|_{L^2(\tau)}$$

$$\sim H^{-2} \sqrt{hH} + 2 \theta \cdot \frac{1}{hH} \cdot \sqrt{hH} + \frac{1}{h^2} \theta^2 \sqrt{hH}$$

$$\sim \frac{h^{1/2}}{H^{3/2}}.$$

\[ \left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\Omega)} \leq \left\| \frac{\partial^2 v}{\partial x^2} \right\|_{L^2(\Omega)} + 2 \left( \frac{\partial^2 v}{\partial x \partial y} \right)_{L^2(\Omega)} + \left\| \frac{\partial^2 v}{\partial y^2} \right\|_{L^2(\Omega)} \]

\[ \sim H^{-2} \sqrt{H} + 2 \theta \cdot \frac{1}{h} \cdot \sqrt{H} + \frac{1}{h^2} \theta^2 \sqrt{H} \]

\[ \sim \frac{1}{H^{3/2}}, \]

\[ \left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\Omega)} \sim h^{-2} \sqrt{h} \sim \frac{H^{1/2}}{h^{3/2}}, \]

\[ \left\| \frac{\partial^2 v}{\partial y^2} \right\|_{L^2(\Omega)} \sim h^{-2} \sqrt{h} \sim \frac{1}{h^{3/2}}. \]

Now, let’s compute \(|v|_{1,\tau}|_r^2\).

\[ |v|_{1,\tau}^2 = \int _\tau ^H \left[ \left( \frac{2ax}{H^2} + \frac{by}{hH} + \frac{d}{H} \right)^2 + \left( \frac{2cy}{h^2} + \frac{bx}{hH} + \frac{e}{h} \right)^2 \right] \, dx \, dy \]

\[ = \int _0 ^H \int _0 ^h \left[ \left( \frac{2ax}{H^2} + \frac{by}{hH} + \frac{d}{H} \right)^2 + \left( \frac{2cy}{h^2} + \frac{bx}{hH} + \frac{e}{h} \right)^2 \right] \, dx \, dy \]

\[ \sim \frac{1}{2} \int _0 ^H \int _0 ^h \left[ \left( \frac{2ax}{H^2} + \frac{by}{hH} + \frac{d}{H} \right)^2 + \left( \frac{2cy}{h^2} + \frac{bx}{hH} + \frac{e}{h} \right)^2 \right] \, dx \, dy \]

\[ = \frac{4a^2}{h^4} \frac{H^2 h}{3} + \frac{b^2}{h^2} \frac{H^3 h}{3} + \frac{d^2}{H^2} \frac{H^2 h}{3} + \frac{4ab}{hH} \frac{H^2 h^2}{4} + \frac{4ad}{hH} \frac{H^2 h^2}{4} + \frac{bd}{H^2} \frac{h^2 H}{2} + \frac{4c^2}{h^4} \frac{H^3 h}{3} + \frac{b^2}{h^2} \frac{H^3 h}{3} + \frac{e^2}{H^2} \frac{H^2 h}{3} + \frac{4bc}{h^2} \frac{h^2 H^2}{4} + \frac{4ec}{h^2} \frac{h^2 H^2}{4} + \frac{2be}{h^2} \frac{H^2 h}{2} \]

\[ \sim \frac{h}{H} + \frac{H}{h} \left( \frac{h}{H} \rightarrow 0 \text{ as } \theta \rightarrow 0 \right) \]

\[ = \frac{H}{h} \]

Hence, \(|v|_{L^2(\tau)} \leq \frac{\sqrt{H}}{\sqrt{h}}|_r^2\). we obtain

\[ \left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L^2(\tau)} \leq \frac{1}{h} |v|_{1,\tau} \leq \frac{1}{H^{3/2}} |v|_{1,\tau}, \quad k = 1, 2, 3. \]

\[ \left\| \frac{\partial^2 v}{\partial y^2} \right\|_{L^2(\tau)} \leq \frac{1}{h \sqrt{H}} |v|_{1,\tau} \leq \frac{1}{H^{3/2}} |v|_{1,\tau}, \quad k = 1, 2, 3. \]

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