SUPER-GEOMETRIC CONVERGENCE OF A SPECTRAL ELEMENT METHOD FOR EIGENVALUE PROBLEMS WITH JUMP COEFFICIENTS

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Abstract

We propose and analyze a \( C^0 \) spectral element method for a model eigenvalue problem with discontinuous coefficients in the one dimensional setting. A super-geometric rate of convergence is proved for the piecewise constant coefficients case and verified by numerical tests. Furthermore, the asymptotical equivalence between a Gauss-Lobatto collocation method and a spectral Galerkin method is established for a simplified model.


Key words: Eigenvalue, Spectral method, Collocation, Galerkin finite element method.

1. Introduction

We often encounter eigenvalue problems with discontinuous coefficients in practice. Examples of such applications may be found in [11]. In this paper, we consider the following one dimensional model problem: Find \((\lambda, u) \in \mathbb{R}^+ \times H^2(−\pi, \pi)\) such that

\[-u''(x) = \lambda c(x) u(x), \quad u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi).
\]

Here \(c(x) \geq c_0 > 0\) is a piecewise constant, or piecewise analytic function. The physics background of this model problem comes from the source-free Maxwell equations describing the transverse-magnetic mode in the one-dimensional periodic media, where the function \(u\) represents the electric field pattern, and the dielectric function \(c(x)\) describes a unit cell from a multilayer structure with \(2\pi\)-periodicity. This model problem was discussed by Min and

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Gottlieb in [11] where \( C^1 \) conforming spectral collocation methods were constructed on two elements over
\[
H^2_{\text{per}}(-\pi, \pi) = \{ v \in H^2(-\pi, \pi) : v(-\pi) = v(\pi), \, v'(-\pi) = v'(\pi) \},
\]
and error bounds of type \( O(p^{-m}) \) were established. Note that the solution of (1.1) belongs to \( C^1 \).

It would be interesting to discuss \( C^0 \) spectral element methods over \( H^1_{\text{per}}(-\pi, \pi) \), since the construction of a \( C^0 \) spectral element method is much simpler than that of the global \( C^1 \) spectral collocation method proposed in [11]. The idea of the spectral element can be found, e.g., in an early work [12]. Note that the spectral element method is equivalent to the so-called \( p \)-version finite element method, see, e.g., [3]. Under the finite element variational framework, we are able to prove a super-geometric error bound of type \( O(e^{-2p(\log p - \gamma)}) \). In some earlier works of the third author, the super-geometric error bound of type \( O(e^{-p(\log p - \gamma)}) \) has been established for some spectral/collocation approximations of the two-point boundary problem [17, 18]. Our error bound for the eigenvalue approximation “doubles” the error bound for the associated eigenfunction approximation, the fact we have known for the \( h \)-version finite element method. It is worthy to point out that in the literature of the spectral method, it is a common practice to consider error bounds of type \( O(p^{-m}) \), see, e.g., [5–7, 10, 15, 16], and reference therein. To the best of our knowledge, this is the first time that a super-geometric convergence rate is established for the eigenvalue approximation by the spectral method.

2. Theoretical Setting

The variational formulation of (1.1) is to find \((\lambda, u) \in \mathbb{R}^+ \times H^1_{\text{per}}(-\pi, \pi)\) such that
\[
(u',v') = \lambda(cu,v), \quad \forall v \in H^1_{\text{per}}(-\pi, \pi).
\]
In this paper, we also consider the Dirichlet problem
\[
-u''(x) = \lambda c(x)u(x), \quad u(0) = 0 = u(1).
\]
Its variational formulation is to find \((\lambda, u) \in \mathbb{R}^+ \times H^1_0(0,1)\) such that
\[
(u',v') = \lambda(cu,v), \quad \forall v \in H^1_0(0,1).
\]

By the general theory [2, 8], both problems (2.1) and (2.2) have countable infinite sequence of eigen-pairs \((\lambda_j, u_j)\) satisfying
\[
0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty, \quad (u_i',u_j') = \lambda_j(cu_i,u_j) = \lambda_j \delta_{ij}.
\]
Furthermore, eigenvalues can be characterized as extrema of the Rayleigh quotient \( R(u) = (u',u')/(cu,u) \) as follows
\[
\lambda_1 = \inf_{u \in S} R(u_1),
\]
\[
\lambda_k = \inf_{u \in S, (u',u_j')=0,j=1,\ldots,k-1} R(u) = R(u_k), \quad k = 2, 3, \ldots,
\]
where \( S = H^1_{per}(-\pi, \pi) \) or \( H^1_0(0, 1) \).

Next, we describe the framework of our numerical approximation. We partition the solution interval into \( m \) sub-intervals (element) such that \( c(x) \) is analytic on each interval. Let \( h \) be the maximum length of all elements, we then define a finite dimensional subspace \( S^h_p \subset S \), as a piecewise polynomial of degree \( p \) on each element. Our spectral element method is to find an eigen-pair \((\lambda(p), w_p) \in R^+ \times S^h_p \) such that

\[
(w'_p, v') = \lambda(p)(cw_p, v), \quad \forall v \in S^h_p.
\]

Note that the partition parameter \( h \) is fixed and convergence is achieved by increasing polynomial degree \( p \). Therefore, we may suppress the index \( h \) later.

By the general theory \([2, 8]\), the problem (2.3) has a finite sequence of eigen-pairs \((\lambda_{j,p}, w_{j,p})\) satisfying

\[
0 < \lambda_{1,p} \leq \lambda_{2,p} \leq \cdots \leq \lambda_{N,p}, \quad N = \left\{ \begin{array}{ll} mp - 1 & H^1_0(0, 1) \\
mp & H^1_{per}(-\pi, \pi) \end{array} \right.
\]

\[
(w'_p, w'_{j,p}) = \lambda_{j,p}(cw_p, w_{j,p}) = \lambda_{j,p}\delta_{ij};
\]

\[
\lambda_{1,p} = \min_{w \in S^h_p} R(w_{1,p}),
\]

\[
\lambda_{k,p} = \min_{w \in S^h_p} \{ R(w) = R(w_{k,p}), \quad k = 2, 3, \ldots \}
\]

One important observation from the above Minimum-Maximum principle is that the specific eigenvalue approximation is from above in the sense

\[
\lambda_k \leq \cdots \leq \lambda_{k,p+1} \leq \lambda_{k,p} \leq \lambda_{k,p-1} \cdots \leq \lambda_{k,1}.
\]

3. A Galerkin Spectral (p-Version) Method

Without loss of generality, we consider piecewise constant \( c(x) \) as in [11] with jump at the center of the solution domain. In particular, we take

\[
c(x) = \left\{ \begin{array}{ll} 1 & x \in (-\pi, 0), \\
\omega^2 & x \in (0, \pi). \end{array} \right.
\]

Instead of constructing \( C^1 \) shape functions for eigenvalue problem (2.1), we seek for a \( C^0 \) approximation \( w_p \in H^1(0, 1) \) with traditional expansion

\[
w_p(x) = w^0(N_- + N_+) + \sum_{j=1}^{p-1} w^j\phi_{p-j+1}(x) + w_pN(x) + \sum_{j=p+1}^{2p-1} w^j\psi_{j-p+1}(x),
\]

where \( N_-(x), N(x), \) and \( N_+(x) \) are linear nodal shape functions at the left end, middle point, and right end of the solution interval, respectively; \( \phi_j \) and \( \psi_j \) are bubble functions on the left and right intervals, respectively. The counterpart of \( \phi_{k+1} \) in \([-1, 1]\) is defined as

\[
\hat{\phi}_{k+1}(\xi) = \sqrt{\frac{2k+1}{2}} \int_{-1}^{\xi} L_k(t) dt = \frac{1}{\sqrt{2(2k+1)}} \left( L_{k+1}(\xi) - L_{k-1}(\xi) \right).
\]

and the counterpart of \( \psi_{k+1} \) in \([-1, 1]\) is defined similarly. Note that \( w^0 = 0 \) for the eigenvalue problem (2.2). With this setting, the resulting stiffness matrix is diagonal and the mass matrix is 5-diagonal, see Appendix.
4. Super-Geometric Convergence Rate

Let \((\lambda_k, u_k)\) be the \(k\)th eigen-pair and \((\lambda_{k,h}, u_{k,h}) \in \mathbb{R} \times S^h\) be its \(h\)-version finite element approximation. According to [2, p.700],

\[ C_1 \epsilon_h^2 \leq \lambda_{k,h} - \lambda_k \leq C_2 \epsilon_h^2, \]

with

\[ \epsilon_h = \inf_{\chi \in S^h} \| u_k - \chi \|_1, \]

for simple eigenvalue \(\lambda_k\) (see [2, p.695 (8.21)]. Transferring this theory to our spectral element method language, we have, for any simple eigen-pair \((\lambda, u)\),

\[ C_1 \epsilon_p^2 \leq \lambda_p - \lambda \leq C_2 \epsilon_p^2, \tag{4.1} \]

with

\[ \epsilon_p = \inf_{\chi \in S^p} \| u - \chi \|_1 \approx \inf_{\chi \in S^p} \| u' - \chi' \| \approx \| u' - u'_p \|, \]

where \(u_p \in S^p\) such that \(u'_p\) is the piecewise Legendre expansion of \(u'\) (not solution of (2.1) or (2.2)). Note that the first “\(\approx\) comes from the Poicaré inequality and the last “\(\approx\) is based on the fact that the Legendre expansion minimizes the \(L^2\)-norm.

Lemma 4.1. Let \(u\) satisfy the regularity assumption

\[
\max_{x \in [-1,1]} |u^{(k)}(x)| \leq cM^k
\]

for fixed constants \(c\) and \(M\), and let \(\tilde{u}'_p\) be the Legendre expansion of \(u'\) on \([-1,1]\). Then under the assumption \((2p+1)!(2p+3) > 2M^2\),

\[
\| u' - \tilde{u}'_p \|_{L^2[-1,1]} \leq C \sqrt{p} \left( \frac{cM}{2p} \right)^{p+1},
\]

where \(C\) is independent of \(p\) and \(M\).

Proof. The error of \((p-1)\)-term Legendre expansion is

\[
\| u' - \tilde{u}'_p \|_{L^2[-1,1]}^2 = \sum_{k=p}^{\infty} \frac{2}{2k+1} b_k^2.
\]

Using the result [13, p.58, Theorem 2.1.6], we have

\[
b_k = \frac{2^k k!}{(2k)!} u^{(k+1)}(\eta_k), \quad \eta_k \in (-1, 1). \tag{4.4}
\]

Note that \((2^k k!)/(2k)! = 1/(2k-1)!!\). Applying the regularity assumption \(|u^{(k)}(x)| \leq cM^k\), we derive

\[
\| u' - \tilde{u}'_p \|_{L^2[-1,1]}^2 < 2(cM^{p+1})^2 \left( \frac{1}{(2p-1)!!(2p+1)!! + \frac{M^2}{(2p+1)!!(2p+3)!! + \frac{M^4}{(2p+3)!!(2p+5)!! + \cdots}} \right)
\]

\[
= \frac{2(cM^{p+1})^2}{(2p-1)!!(2p+1)!!} \left( 1 + \frac{M^2}{(2p+1)(2p+3) + \frac{M^4}{(2p+1)(2p+3)^2(2p+5) + \cdots \right)}
\]

\[
< \frac{4(cM^{p+1})^2}{(2p-1)!!(2p+1)!}, \tag{4.5}
\]
when \((2p + 1)(2p + 3) > 2M^2\). This last term can be readily estimated by Stirling type formula [1, (4.48)]
\[
 n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi \left(n + \frac{1}{6}\right)},
\]
and [4]
\[
 (2n - 1)!! \approx \frac{\sqrt{(2n)!}}{\sqrt{\pi n(n + \frac{1}{4})}},
\]
Therefore,
\[
 (2p - 1)!!(2p + 1)!! \approx \frac{\sqrt{(2p)!}}{\sqrt{\pi (p + 0.25)}} \frac{\sqrt{(2p + 2)!}}{\sqrt{\pi (p + 1.25)}}
\approx \frac{(2p)!2}{\sqrt{\pi p}} \approx \left(\frac{2p}{e}\right) 2\sqrt{2p},
\]
which, combined with (4.5), leads to (4.2) with \(C = \sqrt{2e}\).

Now we are ready to prove our main theorem.

**Theorem 4.2.** Let \((\lambda, u) \in \mathbb{R}^+ \times \mathcal{S}\) be an eigen-pair of problem (2.2), where \(\lambda\) is a simple eigenvalue. Let \(\lambda(p)\) be its approximation in the sense of (2.4) or (2.5). Then
\[
\lambda(p) - \lambda \leq C_p \left(\frac{e\sqrt{\lambda}}{4p}\right)^{2p+2},
\]
where \(C\) is independent of \(p\) and \(M\).

**Proof.** Recall that \(c(x)\) is constants on \((0, 1/2)\) and \((1/2, 1)\), we separate
\[
 \|u' - u'_p\|^2 = \|u' - u'_p\|_{L^2(0, 1/2)}^2 + \|u' - u'_p\|_{L^2(1/2, 1)}^2.
\]
Recall that \(u'_p\) is the piecewise Legendre expansion of \(u'\). The estimate for the first term is as follows,
\[
 \|u' - u'_p\|_{L^2(0, 1/2)}^2 \leq 4\|\hat{u} - \hat{u}'_p\|_{L^2(-1, 1)}^2,
\]
where \(\hat{u}(\xi) = u((1 + \xi)/4)\). Now we apply Lemma 4.1 to have
\[
 \|\hat{u} - \hat{u}'_p\|_{L^2(-1, 1)} \leq C \sqrt{p} \left(\frac{e\sqrt{\lambda}}{2p}\right)^{p+1} = C \sqrt{p} \left(\frac{e\sqrt{\lambda}}{4p}\right)^{p+1}.
\]
Note that
\[
 M = \frac{M}{4} = \frac{\sqrt{\lambda}}{2},
\]
where \(M\) comes from the condition
\[
 \max_{x \in [0, 1]} |u^k(x)| \leq cM^k.
\]
In our situation, \(u\) contains terms like \(\sin 2\sqrt{\lambda}x, \cos 2\sqrt{\lambda}x, \ldots\) \(M = 2\sqrt{\lambda}\), and \(M = 4\hat{M}\) from
\[
 \frac{du}{dx}(x) = 4 \frac{d\hat{u}}{d\xi}(\xi).
\]
The second term in (4.9) can be estimated similarly, and therefore, we have
\[ \|u' - u'_p\| \leq C\sqrt{p} \left( \frac{e\sqrt{\lambda}}{4p} \right)^{p+1} \cdot \]

The error bound (4.8) follows from (4.1).

**Theorem 4.3.** Let \((\lambda, u)\) be an eigen-pair of problem (2.1), where \(\lambda\) is a simple eigenvalue. Let \(\lambda(p)\) be its approximation in the sense of (2.4) or (2.5). Then
\[ \lambda(p) - \lambda \leq Cp \left( \frac{e\pi\sqrt{\lambda}}{2p} \right)^{2p+2}. \] (4.11)

**Proof.** The proof is the same by the scaling between \((0, 1)\) and \((-\pi, \pi)\).

**Remark.** The error bounds in Theorems 4.2 and Theorem 4.3 are super-geometric type \(O(e^{-\gamma p})\) with \(\gamma = \ln(e\sqrt{\lambda}/4)\) and \(\gamma = \ln(e\pi\sqrt{\lambda}/2)\), respectively. We shall demonstrate in the next section, by numerical tests, that our error bounds are sharp. Our estimates also indicate that we need higher \(p\) for larger \(\lambda\) to realize the convergence. This is consistent with our numerical experiences.

5. **Numerical Tests**

In this section, we implement the numerical scheme described in Section 3 to solve (1.1) with \(\omega = 2\) for the first 14 eigenvalues. We observe convergence for reasonably smaller \(p\). Actually, the error goes to the machine \(\epsilon\) for \(p \leq 10\) for the first few eigenvalues. To verify our error bounds, we plot the ratio
\[ \frac{(\lambda(p) - \lambda)}{p(0.5e\pi\sqrt{\lambda}/p)^{2p+2}} \] (5.1)
with some different \(\lambda\). Here is a list of the square roots of eigenvalues (\(\sqrt{\lambda}\) by increasing order):

\[
\begin{align*}
\frac{1}{\pi} \arccos \left( -\frac{1}{3} \right), & \quad \frac{1}{\pi} \arccos \left( -\frac{2}{3} \right), & \quad \frac{1}{\pi} \arccos \left( \frac{2}{3} \right) + 1, & \quad \frac{1}{\pi} \arccos \left( \frac{1}{3} + 1 \right), & \quad 2, \\
\frac{1}{\pi} \arccos \left( -\frac{1}{3} \right) + 2, & \quad \frac{1}{\pi} \arccos \left( -\frac{2}{3} \right) + 2, & \quad \frac{1}{\pi} \arccos \left( \frac{2}{3} \right) + 3, & \quad \frac{1}{\pi} \arccos \left( \frac{1}{3} + 3 \right), & \quad 4, \\
\frac{1}{\pi} \arccos \left( -\frac{1}{3} \right) + 4, & \quad \frac{1}{\pi} \arccos \left( -\frac{2}{3} \right) + 4, & \quad \frac{1}{\pi} \arccos \left( \frac{2}{3} \right) + 5, & \quad \frac{1}{\pi} \arccos \left( \frac{1}{3} + 5 \right), & \quad 6, \ldots
\end{align*}
\]

Figures 1-4 demonstrate the ratio (5.1) associated with \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_9, \lambda_{14}\), respectively. We plot the ratio with different range of \(p\). Since for a larger eigenvalue, we need relatively higher \(p\) to get into the asymptotic range. On the other hand, when \(p\) getting bigger and the error approaching the machine \(\epsilon\), the round-off error kicks in. So we can only observe the ratio in a small range of \(p\). Nevertheless, it is sufficient to make our point clear. We see that the ratio (5.1) maintains in a reasonable range for different eigenvalues.

6. **A Collocation Method for the Smooth Case**

As a special case when \(c(x)\) is sufficiently smooth, say a constant, we may use only one element. Without loss of generality, let us consider
\[ -u'' = \lambda u \quad \text{in} \quad (-1, 1) \quad u(-1) = 0 = u(1). \] (6.1)
In this case, we are seeking an eigen-pair \((\lambda(p), w_p)\) with

\[ w_p(\xi) = \sum_{j=2}^{p} w_j \hat{\phi}_j(\xi) \]

to satisfy

\[ (w'_p, \hat{\phi}_k) = \lambda(p)(w_p, \hat{\phi}_k), \quad k = 2, \ldots, p. \]  \hspace{1cm} (6.2)

Again, this result in an identity matrix on the left and a 5-diagonal matrix on the right. Based on the analysis in Section 4, we have in this case

\[ \lambda(p) - \lambda \leq C p \left( \frac{e \sqrt{\lambda}}{2p} \right)^{2p+2}. \]  \hspace{1cm} (6.3)

Let us consider a spectral collocation method

\[ -w''_p(x_j) = \lambda(p)w_p(x_j), \quad j = 1, 2, \ldots, p - 1, \]  \hspace{1cm} (6.4)

where \(x_j\)s are zeros of \(L'_p\) and \(L_p\) is the Legendre polynomial of degree \(p\) on \([-1, 1]\).

**Theorem 6.1.** For the model problem (6.1), the spectral collocation method (6.4) is equivalent to replacing all integrations in (6.2) by the \(p + 1\)-point Gauss-Lobatto quadrature. Furthermore, all numerical integrations are exact except the last term with

\[ (\hat{\phi}_p, \hat{\phi}_p)^* = \sum_{j=0}^{p} \hat{\phi}_p^2(x_j)w_j = (\hat{\phi}_p, \hat{\phi}_p) \frac{3(2p^2 - p - 1)}{2(2p^2 - p)}, \]  \hspace{1cm} (6.5)

where \(w_j\)s are weights of the Gauss-Lobatto quadrature.

**Proof.** We multiply both sides of (6.4) by \(\hat{\phi}_k(x_j)w_j\) and sum up

\[ -\sum_{j=0}^{p} w''_p(x_j)\hat{\phi}_k(x_j)w_j = \lambda(p) \sum_{j=0}^{p} w_p(x_j)\hat{\phi}_k(x_j)w_j \]  \hspace{1cm} (6.6)

Since the \(p + 1\)-point Gauss-Lobatto quadrature rule is exact for polynomials of degree up to \(2p - 1\), then we have

\[ -\sum_{j=0}^{p} w''_p(x_j)\hat{\phi}_k(x_j)w_j = -(w''_p, \hat{\phi}_k) = (w'_p, \hat{\phi}_k), \quad k = 2, \ldots, p; \]

\[ \sum_{j=0}^{p} w_p(x_j)\hat{\phi}_k(x_j)w_j = (w_p, \hat{\phi}_k), \quad k = 2, \ldots, p - 1. \]
We see that the collocation method (6.4) is almost identical to the spectral method (6.2) except one term

\[(w_p, \hat{\phi}_p) \neq \sum_{j=0}^{p} w_p(x_j)\hat{\phi}_p(x_j)w_j = (w_p, \hat{\phi}_p)_*,\]

and their difference is

\[(w_p, \hat{\phi}_p) - (w_p, \hat{\phi}_p)_* = w^p[(\hat{\phi}_p, \hat{\phi}_p) - \hat{\phi}_p, \hat{\phi}_p)_*].\]
Using the fact \((L_p, L_p)_* = 2/p\), a direct calculation yields,
\[
(\hat{\phi}_p, \hat{\phi}_p) = \frac{1}{2(2p - 1)}(L_p - L_{p-2}, L_p - L_{p-2})
\]
\[
= \frac{1}{2(2p - 1)}((L_p, L_p) + (L_{p-2}, L_{p-2}))
\]
\[
= \frac{1}{2(2p - 1)} \left( \frac{2}{2p + 1} + \frac{2}{2p - 3} \right),
\] (6.7)
and
\[
(\hat{\phi}_p, \hat{\phi}_p)_* = \frac{1}{2(2p - 1)}((L_p, L_p)_* + (L_{p-2}, L_{p-2}))
\]
\[
= \frac{1}{2(2p - 1)} \left( \frac{2}{p} + \frac{2}{2p - 3} \right).
\] (6.8)
Therefore,
\[
\frac{(\hat{\phi}_p, \hat{\phi}_p)_*}{(\phi_p, \phi_p)} = \frac{3(2p + 1)(p - 1)}{2p(2p - 1)},
\]
and (6.5) follows.

We see that the \(p + 1\)-points Gauss-Lobatto quadrature has a 50% over-shoot asymptotically in calculating \((\hat{\phi}_p, \hat{\phi}_p)_*\). Nevertheless, the last coefficient \(w^p\) decays fast in general and the collocation method (6.4) is asymptotically equivalent to the spectral method (6.2). As a consequence, the spectral collocation method (6.4) also enjoys the super-geometric convergence rate (6.3).

**Remark.** It is feasible that a parallel result may be developed for the Chebysheve spectral/collcocation methods. It is also feasible that the error bounds in [11] may be improved to the similar super-geometric rate as in this paper.

7. Appendix. The property of stiffness matrix

Since \((\hat{\phi}_i, \hat{\phi}_j) = \delta_{ij}\), it is straightforward to verify that
\[
(\phi_i', \phi_j') = 4\delta_{ij} = (\psi_i', \psi_j'), \quad (N', \phi_j') = 0 = (N', \psi_j'), \quad (N', N') = 4.
\]
Furthermore, observe that
\[
4(\phi_1, \phi_j) = (\hat{\phi}_1, \hat{\phi}_j) = 4(\psi_1, \psi_j),
\]
\[
(\phi_{k+1}, N) = 0 = (\psi_{k+1}, N), \quad k > 2,
\]
\[
(N, N) = \frac{1}{4} \int_{-1}^{1} \left( \frac{1 + \xi}{2} \right)^2 + \left( \frac{1 - \xi}{2} \right)^2 = \frac{1}{16} \left( 4 + \frac{4}{3} \right) = \frac{1}{3},
\]
\[
(\phi_2, N) = \frac{1}{4} \int_{-1}^{1} \left( L_2 - L_0, \hat{N}_2 \right) = \frac{-1}{4\sqrt{6}} \int_{-1}^{1} \frac{1 + \xi}{2} = \frac{-1}{4\sqrt{6}},
\]
\[
(\phi_3, N) = \frac{1}{4} \left( L_3 - L_1, \hat{N}_2 \right) = \frac{-1}{4\sqrt{10}} \int_{-1}^{1} \frac{1 + \xi}{2} = \frac{-1}{4\sqrt{10}},
\]
\[
(\psi_2, N) = \frac{1}{4} \left( L_2 - L_0, \hat{N}_1 \right) = \frac{-1}{4\sqrt{6}} \int_{-1}^{1} \frac{1 - \xi}{2} = \frac{-1}{4\sqrt{6}},
\]
\[
(\psi_3, N) = \frac{1}{4} \left( L_3 - L_1, \hat{N}_1 \right) = \frac{-1}{4\sqrt{10}} \int_{-1}^{1} \frac{1 - \xi}{2} = \frac{1}{12\sqrt{10}}.
\]
The special arrangement of ordering has advantage of symmetric structure in the resulting
matrix. It is straightforward to verify that the weak formulation

\[(u', v') = \lambda (cu_p, v), \quad \forall v \in S_p\]

yields a diagonal stiffness matrix \(4I\) and a 5-diagonal mass matrix \(\frac{1}{4}A\), where the upper half of
\(A\) is of the form

\[
\begin{pmatrix}
\frac{2}{(2p+1)(2p-3)} & 0 & \frac{2}{(2p-1)(2p-5)} & \frac{1}{(2p-3)(2p-7)} & \cdots \\
0 & \frac{2}{(2p-1)(2p-5)} & 0 & \frac{2}{(2p-3)(2p-7)} & \cdots \\
\frac{1}{(2p-3)(2p-5)} & 0 & \frac{2}{(2p-3)(2p-7)} & \cdots \\
\cdots & \cdots & \cdots \cdots & \cdots & \\
0 & \frac{2}{45} & 0 & \frac{1}{5\sqrt{21}} & \\
\end{pmatrix}
\]

and the lower half of \(A\) is obtained by taking the above expression up-side-down and multiplied
by \(\omega^2\). The bottom right entry 2/3 of the upper part and the top left entry \(2\omega^2/3\) of the lower
part is the only overlap of the two parts.

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