1(P.230:3(a)). Evaluate the integral
\[ \int_C \frac{z^5}{1 - z^3} dz. \]
Here \( C \) is the circle \( |z| = 2 \) directed positively.

**Answer:** Since all the 3 singular points of the integrant are in the interior of \( C \), we’d better use the theorem on page 228 to write
\[ \int_C \frac{z^5}{1 - z^3} dz = 2\pi i \text{Res}_{z=0} \frac{1}{z^2} \frac{(1/z)^5}{1 - (1/z)^3} = -2\pi i. \]
The reason is that this is equal to \( 2\pi i \text{Res}_{z=0} \frac{1}{z^4} \frac{1}{z^3 - 1} \). We use the standard expansion \( \frac{1}{z^3 - 1} = -\sum_{n=0}^{\infty} z^{3n} \) to determine that the \( c_{-1} \) coefficient in the Laurent expansion of \( \frac{1}{z^4(z^3 - 1)} \) is \(-1\), which is the desired residual.

2(P.234:4). Write the function \( f(z) = \frac{8a^3z^2}{(z^2 + a^2)^3} = \frac{\phi(z)}{(z - ai)^3} \), with \( \phi(z) = \frac{8a^3z^2}{(z + ai)^3} \). Point out why \( \phi \) has a Taylor expansion about \( ai \), and then use it to show that the principal part of \( f \) at the point is
\[ \frac{\phi''(ai)/2}{z - ai} + \frac{\phi'(ai)}{(z - ai)^2} + \frac{\phi(ai)}{(z - ai)^3} = \frac{i/2}{z - ai} - \frac{a/2}{(z - ai)^2} - \frac{a^2i}{(z - ai)^3}. \]

**Answer:** The function \( \phi \) is analytic at \( ai \), since its singularities are at \(-ai\), so \( \phi \) has a Taylor expansion about \( ai \). The principal part in the Laurent expansion of \( f \) then must be the one given in the problem. One needs to calculate the derivatives of \( \phi \) at \( ai \) to see the equation.

3(P.238:3). Find the value of the integral
\[ \int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz, \]
taken counterclockwise around the circle (a) \( |z - 2| = 2 \); (b) \( |z| = 4 \). **Answer:** The integrant \( f \) has singularities at 1 and \( \pm 3i \). For part (a), only 1 is in the interior of \( C \), so
\[ \int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz = 2\pi i \text{Res}_{z=1} f(z) = 2\pi i \cdot \frac{1}{2} = \pi i. \]
For part (b), all the 3 singular points are in the interior of \( C \), therefore,
\[ \int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz = 2\pi i \text{Res}_{z=1,z=\pm 3i} f(z) = 2\pi i \cdot \left( \frac{1}{2} + \frac{150}{60} \right) = 6\pi i. \]
4(P.245:4(b)). Let $C$ denote the positively oriented circle $|z| = 2$, and evaluate the integral
\[
\int_C \frac{dz}{\sinh 2z}.
\]

**Answer:** The sinh $2z$ function has zeros at $z = \pm \frac{k}{2} \pi i$, of which $0, \pm \frac{\pi}{2} i$ are in the interior of the contour. We need to calculate the residuals of $\frac{1}{\sinh 2z}$ at all the three points. These are the values of $\frac{1}{2 \cosh 2z}$ at these points. They are $\frac{1}{2}, -\frac{1}{2},$ and $-\frac{1}{2}$, the sum of which is $-\frac{1}{2}$. Therefore
\[
\int_C \frac{dz}{\sinh 2z} = -\pi i.
\]

5(P.257:7). Use the residues to find the Cauchy principal value of the integral
\[
\int_{-\infty}^{\infty} \frac{x}{(x^2 + 1)(x^2 + 2x + 2)} dx.
\]

**Answer:** We use the close contour integral $\int_{-R}^{R} + \int_{C_R}$, with $C_R$ being the upper semicircle of radius $R$. In the upper half plane, there are two singularities of the complexified integrant at $i$ and $1+i$. We choose $R$ so big that encloses both of these. Therefore $\int_{-R}^{R} + \int_{C_R} = 2\pi i \text{Res}_{z=i,z=1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)}$. We write
\[
\frac{z}{(z^2 + 1)(z^2 + 2z + 2)} = \frac{z}{(z+i)(z+1+i)(z+1-i)},
\]
The residue at $i$ is the value of $\frac{z}{(z+i)(z+1+i)(z+1-i)}$ there. It is $\frac{1-2i}{10}$. Similarly, the residue at $-1+i$ is $\frac{-1+3i}{10}$. Therefore, $\int_{-R}^{R} + \int_{C_R} = -\frac{\pi}{5}$. Then, we note that $\int_{C_R} \to 0$ as $R \to \infty$.

6(P.265:8). Use the residues to find the value of
\[
\int_{0}^{\infty} \frac{x^3 \sin x}{(x^2 + 1)(x^2 + 9)} dx.
\]

**Answer:** First, the integrant is even, so the value of the integral is half of the integral on the whole real number line. We consider the complex integral
\[
\int_{-\infty}^{\infty} \frac{z^3 e^{iz}}{(z^2 + 1)(z^2 + 9)} dz,
\]
of which the imaginary part gives us the desired result. When $R$ is sufficiently big, we have
\[
\int_{-R}^{R} + \int_{C_R} = 2\pi i \text{Res}_{z=\pm i,3i} \frac{z^3 e^{iz}}{(z^2 + 1)(z^2 + 9)} = \frac{27e^{-3} - 3e^{-1}}{24} \pi i.
\]
We note that $\int_{C_R} \to 0$ as $R \to \infty$. Thus
\[
\int_{0}^{\infty} \frac{x^3 \sin x}{(x^2 + 1)(x^2 + 9)} = \frac{27e^{-3} - 3e^{-1}}{48} \pi.
\]
7(P.277:6). Show that

\[
\int_0^\infty \frac{dx}{\sqrt{x(x^2+1)}} = \frac{\pi}{\sqrt{2}}
\]

**Answer:** We approach this by using the complex function \(\frac{1}{z^{1/2}(z^2+1)}\). Besides the singularity due to a branch cut associated with \(z^{1/2}\), there are two isolated singularities at \(\pm i\), respectively. We could use two methods to solve this. One is based on an indented upper semicircle as the integral contour. (See figure 97 in the book.) We could use the negative \(y\)-axis as the branch cut, which only touch our semicircle at the origin, which is excluded by using a small semicircle \(C_{\epsilon}\). Then

\[
\int_C \frac{1}{z^{1/2}(z^2+1)} \, dz = 2\pi i \text{Res}_{z=i} \frac{1}{z^{1/2}(z^2+1)} = 2\pi i \frac{1}{e^{i\pi/4}2i} = \frac{\sqrt{2}\pi(1-i)}{2}.
\]

We also have

\[
\int_{-\epsilon}^{-\infty} \frac{1}{z^{1/2}(z^2+1)} \, dz = \int_{-\epsilon}^{-\infty} \frac{1}{\sqrt{|x|e^{i\pi/2}(x^2+1)} \, dx = -i \int_{\epsilon}^{R} \frac{1}{\sqrt{x(x^2+1)}} \, dx.
\]

We then let \(\epsilon \to 0\) and \(R \to \infty\), argue that both the \(C_{\epsilon}\) and \(C_{R}\) integrals converge to zero, and get

\[
(1-i) \int_0^\infty \frac{dx}{\sqrt{x(x^2+1)}} = \frac{\sqrt{2}\pi(1-i)}{2}.
\]

Thus \(\int_0^\infty \frac{dx}{\sqrt{x(x^2+1)}} = \frac{\sqrt{2}\pi}{2}\).

One could also choose the closed contour as depicted in Figure 99.

8(P.280:5). Find the integral

\[
\int_{0}^{\pi} \frac{\cos 2\theta d\theta}{1-2a \cos \theta + a^2} \quad (-1 < a < 1).
\]

**Answer:** This is equal to one half of the integral on the interval \((-\pi, \pi)\). By changing variable \(e^{\theta i} = z\), using the fact that \(\cos \theta = (z + z^{-1})/2\), \(\cos 2\theta = (z^2 + z^{-2})/2\), and \(d\theta = dz/iz\), we can write the latter integral as a closed contour integral on the unit circle \(C\) in the \(z\)-plane. Thus

\[
\int_{0}^{\pi} \frac{\cos 2\theta d\theta}{1-2a \cos \theta + a^2} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos 2\theta d\theta}{1-2a \cos \theta + a^2}
\]

\[
= -\frac{1}{4ai} \int_{C} \frac{z^4 + 1}{z^2(z-a)(z-\frac{1}{a})} \, dz = -\frac{1}{4ai} 2\pi i \text{Res}_{z=0,z=a} \frac{z^4 + 1}{z^2(z-a)(z-\frac{1}{a})} = \frac{a^2\pi}{1-a^2}.
\]

Note that the residue at 0 is equal to the derivative of \(\frac{z^4+1}{(z-a)(z-\frac{1}{a})}\) at 0, and the residue at \(a\) is just the value of \(\frac{z^4+1}{z^2(z-\frac{1}{z})}\) at \(z = a\).
9(P.286:8). Determine the number of rppts, counting multilicity, of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \leq z < 2$.

Answer: We determine the number of zeros in $|z| < 2$ and in $|z| < 1$, and to get the answer by minus the latter from the former. We use Rouché's theorem. To resolve the issue on $|z| < 2$, we consider taking $f(z) = 2z^5$ and $g(z) = -6z^2 + z + 1$. Since $|g(z)| < |f(z)|$ for $|z| = 2$, the number of zeros in $|z| < 2$ of $f + g$ is the same as that of $f$, which is 5.

To resolve the problem on $|z| < 1$, we let $f(z) = -6z^2$ and $g(z) = 2z^5 + z + 1$ to verify the condition of Rouché's theorem. Thus there are 2 zeros. The number of zeros in the annulus is then 3.

10(P.286:9). Show that if $c$ is a complex number such that $|c| > e$, then the equation $cz^n = e^z$ has $n$ roots, counting multiplicity, inside the circle $|z| = 1$.

Answer: A root of the equation $cz^n = e^z$ is a zero of the function $cz^n - e^z$. Let $f(z) = cz^n$ and $g(z) = -e^{z}$. Since $|c| > e$, we have $|g(z)| < |f(z)|$ for $|z| = 1$. Thus, according to Rouché, in $|z| < 1$, the number of zeros of $f + g$ is the same as that of $f$, which is $n$. 