An Adams Spectral Sequence Primer

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Abstract. My aim with these notes is to quickly get the student started with the Adams spectral sequence. To see its power requires that some concrete calculations be done. However, the algebra required can quickly become overwhelming if one starts with the generalized Adams spectral sequence. The classical Adams spectral sequence, in contrast, can be quickly set up and used to do some calculations which would be quite difficult by any other technique. Further, the classical Adams spectral sequence is still a useful calculational and theoretical tool, and is an excellent introduction to the general case.

Throughout, I have tried to motivate the ideas with historical remarks, heuristic explanations, and analogy. For prerequisites, the reader should be willing to work in some category of spectra, though little more than the notion of stabilizing the homotopy types of spaces will really be needed. In particular, no familiarity with the details of any particular category of spectra will be assumed. A passing acquaintance with calculations with the Steenrod algebra will be helpful, though the details will be provided where they are needed, so a determined novice has a good chance of success.

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CHAPTER 0

History and Introduction.

The Adams spectral sequence is a way of describing the relation between homotopy and homology. Given

(1) spaces or spectra X and Y, and
(2) a cohomology theory E^* (or homology theory E_*)

there is a filtration of $[X, Y]$, the stable homotopy classes of maps from X to Y,

(1) whose filtration quotients are the E_∞ term of a spectral sequence, and
(2) whose E_2 term can be computed from E^*X and E^*Y regarded as $E^*E = \mathrm{End}(E)$-modules (respectively, from E_*X and E_*Y regarded as E_*E-comodules).

Precisely, we have

$$\mathrm{Ext}^{s,t}_{E^*E}(E^*Y, E^*X) \Longrightarrow [X, Y]^E_{t-s}$$

and

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*X, E_*Y) \Longrightarrow [X, Y]^E_{t-s}$$

where $[X, Y]^E$ denotes maps from X to Y in an E-localization. Of course, we must address questions of convergence, but heuristically, this is what the Adams spectral sequence is trying to do. The cohomological version is simpler to construct, but requires stronger assumptions in order to identify E_2 or prove convergence. Since these assumptions are satisfied in many interesting cases, we shall mainly concern ourselves with the cohomological version.

0.1. The d and e invariants

The first two layers of the filtration have a very simple description generalizing the degree of a map and Adams' e-invariant.

$$\begin{array}{rcccl} F_0 & = & [X, Y] & \xrightarrow{d} & \mathrm{Hom}(E^*Y, E^*X) \\ & & \cup & & \\ F_1 & = & \mathrm{Ker}(d) & \xrightarrow{e} & \mathrm{Ext}^1(E^*Y, E^*X) \end{array}$$

Here, $d(f) = f^* : E^*Y \longrightarrow E^*X$, so that if $E = H\mathbb{Z}$ is integral cohomology, and $X = Y = S^n$, or more generally, if X and Y are n-manifolds, then $d(f) \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ is the usual degree of the map f. If $f \in \mathrm{Ker}(d)$, then the long exact sequence induced by f in E-cohomology becomes a short exact sequence and

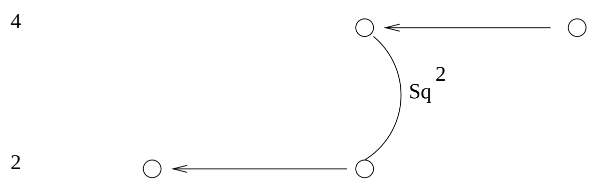
$$e(f) = \{0 \longleftarrow E^*Y \longleftarrow E^*Cf \longleftarrow E^*X \longleftarrow 0\} \in \mathrm{Ext}^1(E^*Y, E^*X),$$

where $Cf = Y \cup_f CX$ is the cofiber of f. (We are suppressing the suspension of X in the short exact sequence above, because we shall need to use homomorphisms of nonzero degree soon enough; we might as well start now.)

For example, the Hopf map $\eta : S^3 \longrightarrow S^2$ induces the short exact sequence in mod 2 cohomology

$$\{0 \longleftarrow H^*S^2 \longleftarrow H^*CP^2 \longleftarrow H^*S^4 \longleftarrow 0\} \in \mathrm{Ext}^{1,0}(H^*S^2, H^*S^4)$$

which we may describe pictorially by



Here, each dot represents an $\mathbb{F}_2$, the curved vertical line represents the action of the Steenrod algebra, $\mathcal{A} = H\mathbb{F}_2^* H\mathbb{F}_2$, and the horizontal arrows represent the action of the homomorphisms. Since this is a nontrivial extension of $\mathcal{A}$-modules, η must be stably essential. We say that this element of Ext^1 *detects* η. To proceed further, we must become more systematic, but first we would like to describe the historical context in which the Adams spectral sequence was invented.

0.2. Computing homotopy via Postnikov towers

At the time Adams introduced the Adams spectral sequence, the main technique for deducing homotopy groups from cohomology was through Postnikov towers. The idea is simple: the Hurewicz theorem allows us to deduce the bottom nonzero homotopy group from the homology or cohomology. We 'kill' this by taking the fiber of a map to a $K(\pi, n)$ which maps this group isomorphically. Such a map can be obtained by attaching cells to the original space to kill all the homotopy above the bottom, thereby producing a $K(\pi, n)$, and taking the map to be the inclusion of our original space into the new one. We then use the Serre spectral sequence to deduce the cohomology of the fiber and repeat the process. We prefer cohomology to homology for this because we find it easier to work with modules over the Steenrod algebra than with comodules over the dual of the Steenrod algebra. This means that we must use the Universal Coefficient Theorem to deduce the homology from the cohomology, and then use the Hurewicz theorem to determine the bottom nonzero homotopy group, so that we may repeat the process.

We also prefer to work with homology and cohomology with field coefficients as far as possible, and deduce the integral homology only when we are about to apply the Hurewicz theorem.

Let us apply this method to the sphere, and let us localize at 2 for simplicity. We will need the mod 2 cohomology of $K(\mathbb{Z}/2, n)$ and $K(\mathbb{Z}, n)$. These were computed in [**14**] in 195x (Cartan seminar?): $H^*K(\mathbb{Z}/2, n)$ is the polynomial algebra on the set $\{Sq^I(\iota_n)\}$, where Sq^I is an admissable Steenrod operation of excess less than n, and $H^*K(\mathbb{Z}, n)$ is the polynomial algebra on the subset of those $Sq^I(\iota_n)$ where $I = (i_1, \ldots, i_k)$ has $i_k > 1$. Here, $Sq^I(\iota_n)$ means $Sq^{i_1}Sq^{i_2}\cdots Sq^{i_k}(\iota_n)$.

We know that $\pi_n S^n$ is the first nonzero homotopy group since it is the only nonzero (reduced) homology group. Thus $H^n S^n = \mathbb{Z}$ as well, and we let $X_1 = \mathrm{Fiber}(S^n \longrightarrow K(\mathbb{Z}, n))$,

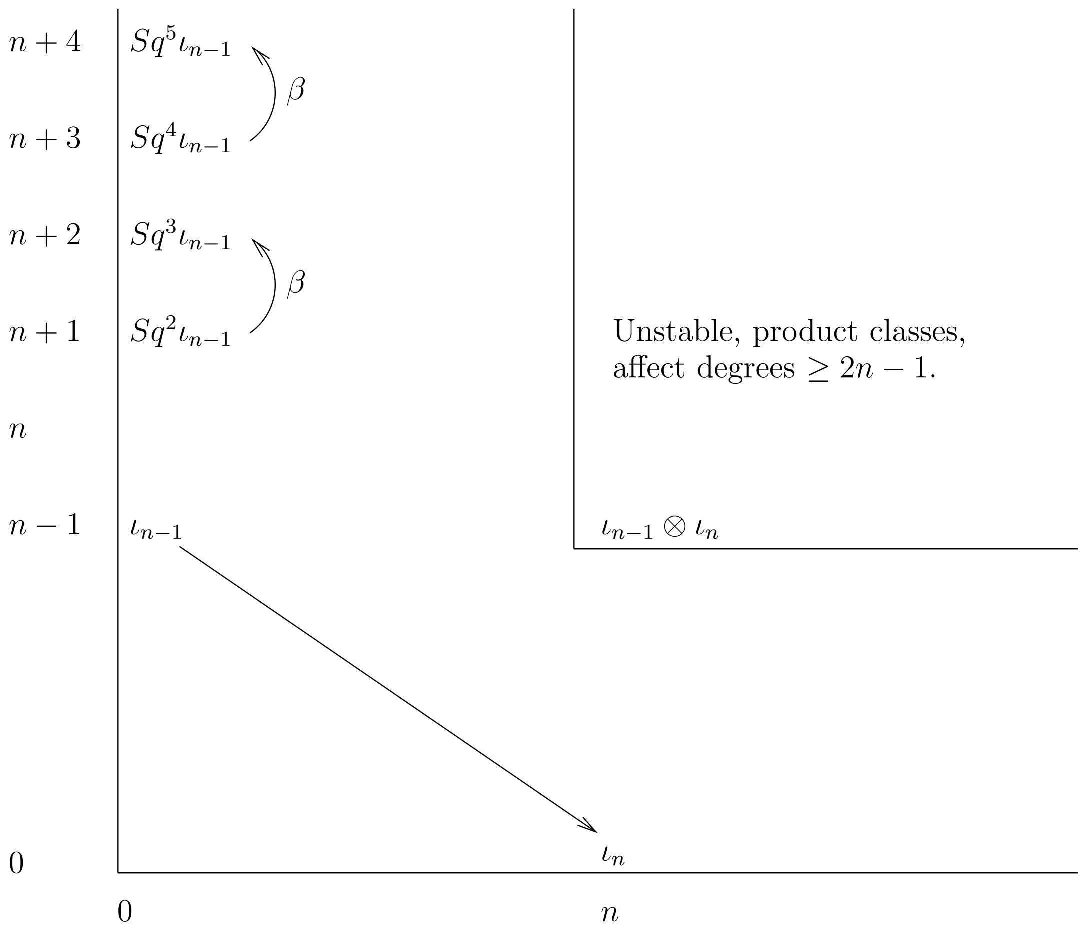


FIGURE 0.1. The cohomology Serre spectral sequence for the fibration $K(\mathbb{Z}, n-1) \longrightarrow X_1 \longrightarrow S^n$

where the map from S^n to $K(\mathbb{Z}, n)$ is an isomorphism on π_n, representing a generator of $H^n S^n$. The long exact sequence in homotopy shows that $\pi_i X_1 \cong \pi_i S^n$ for $i > n$ and $\pi_i X_1 = 0$ for $i \leq n$. To compute the homotopy groups of X_1, we start by computing its cohomology. We have a fibration

$$K(\mathbb{Z}, n-1) \longrightarrow X_1 \longrightarrow S^n$$

whose cohomology Serre spectral sequence has $E_2 = H^* K(\mathbb{Z}, n-1) \otimes H^* S^n$ and one differential $d_n(\iota_{n-1} \otimes 1) = 1 \otimes \iota_n$. (See Figure 0.1.) Let us assume that $2n - 1 > n + 4$, so that the class $\iota_{n-1} \otimes \iota_n$ does not affect the results in dimensions less than $n + 4$. The E_∞ term then tells us that

$$H^i X_1 = \begin{cases} 0 & i < n+1 \\ < Sq^2\iota > & i = n+1 \\ < Sq^3\iota > & i = n+2 \\ < Sq^4\iota > & i = n+3 \\ < Sq^5\iota > & i = n+4 \end{cases}$$

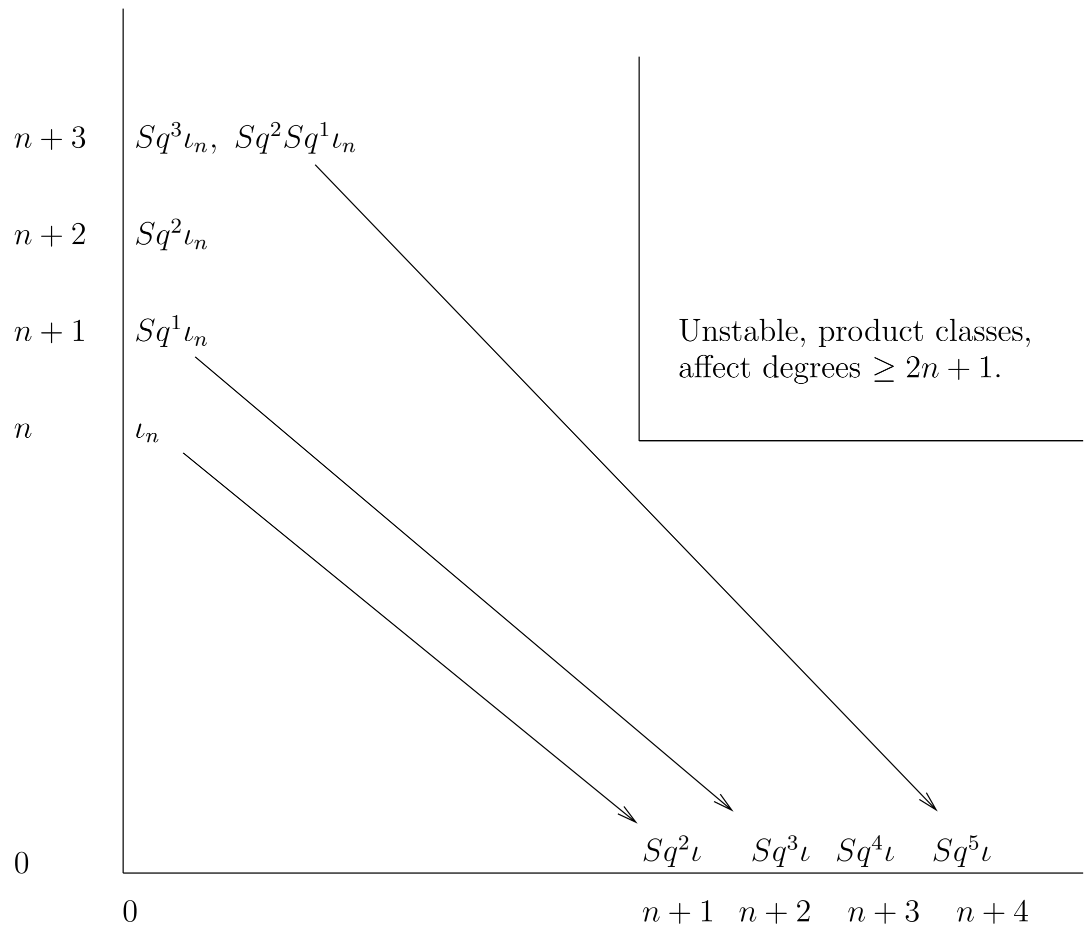


FIGURE 0.2. The cohomology Serre spectral sequence for the fibration $K(\mathbb{Z}/2, n) \longrightarrow X_2 \longrightarrow X_1$

Note that this is simply the kernel of the homomorphism $H^*K(\mathbb{Z}, n) \longrightarrow H^*S^n$ shifted down one dimension. Since the mod 2 cohomology Bockstein $\beta = Sq^1$, and since $Sq^1Sq^2 = Sq^3$, we see that the mod 2 homology Bockstein $\beta : H_{n+2}(X_1; \mathbb{Z}/2) \longrightarrow H_{n+1}(X_1; \mathbb{Z}/2)$ is an isomorphism, and hence $H_{n+1}(X_1; \mathbb{Z}) = \mathbb{Z}/2$. Thus, $\pi_{n+1}S^n = \pi_{n+1}X_1 = Z/2$.

Let $X_2 = \text{Fiber}(X_1 \longrightarrow K(\mathbb{Z}/2, n+1))$, where the map from X_1 to $K(\mathbb{Z}/2, n+1)$ is an isomorphism on π_{n+1}, representing a generator of $H^{n+1}X_1$. Taking the fiber again, we have a fibration

$$K(\mathbb{Z}/2, n) \longrightarrow X_2 \longrightarrow X_1$$

whose cohomology Serre spectral sequence has E_2 term $H^*K(\mathbb{Z}/2, n) \otimes H^*X_1$. The first differential must be $d_{n+1}(\iota_n) = Sq^2(\iota)$ since X_2 was obtained by killing the bottom homotopy group of X_1. Since the transgression commutes with Steenrod operations, we also get that $d_{n+2}(Sq^1(\iota_n)) = Sq^3(\iota)$ and that $d_{n+3}(Sq^2(\iota_n)) = 0$, since $Sq^2Sq^2 = Sq^3Sq^1$ so that $Sq^2(Sq^2(\iota)) = 0$ in X_1. Also, $d_{n+4}(Sq^3(\iota_n)) = 0$ and $d_{n+4}(Sq^2Sq^1(\iota_n)) = Sq^5(\iota)$ since

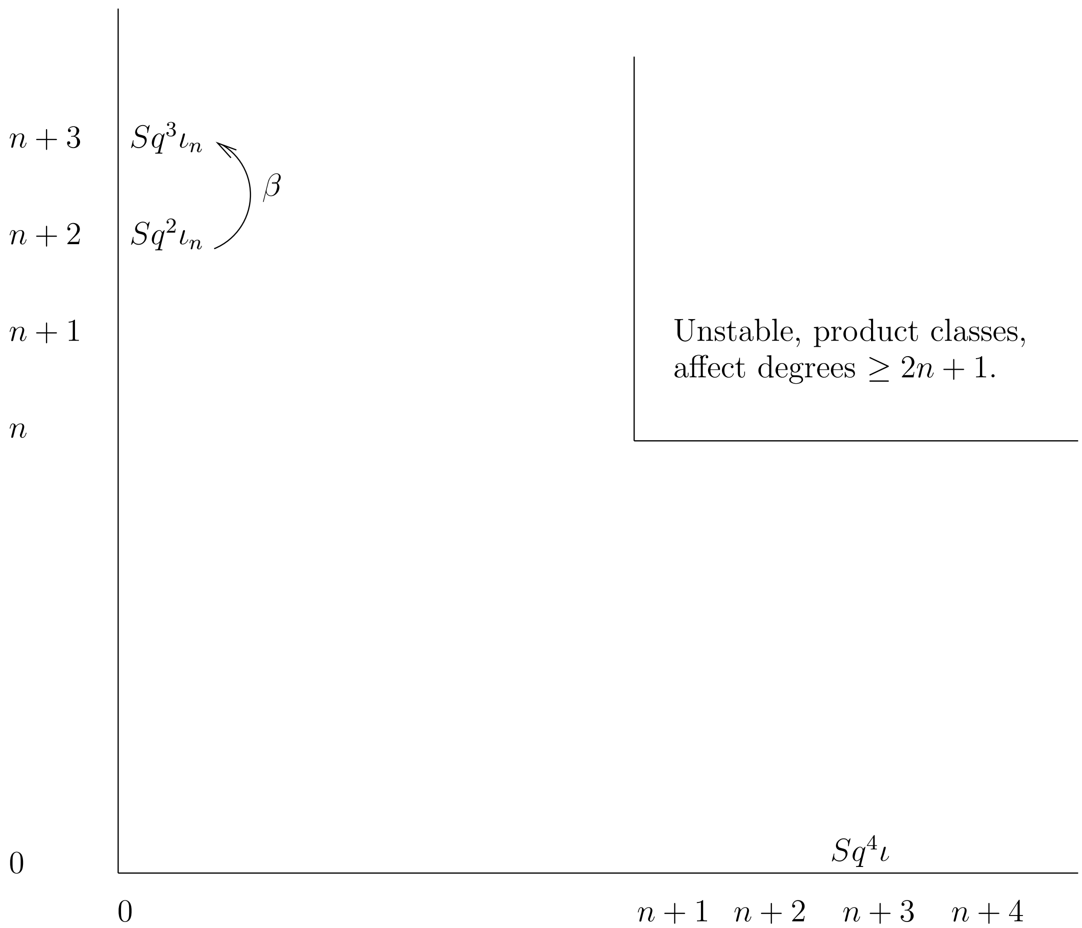


FIGURE 0.3. E_∞ term of the cohomology Serre spectral sequence for the fibration $K(\mathbb{Z}/2, n) \longrightarrow X_2 \longrightarrow X_1$

$Sq^2Sq^1Sq^2(\iota) = Sq^5(\iota) + Sq^4Sq^1(\iota) = Sq^5(\iota)$. This gives

$$H^iX_2 = \begin{cases} 0 & i < n+2 \\ \langle Sq^2\iota_n \rangle & i = n+2 \\ \langle Sq^3\iota_n \rangle \oplus \langle Sq^4\iota \rangle & i = n+3 \end{cases}$$

Again we have a nontrivial cohomology Bockstein $H^{n+2}X_2 \longrightarrow H^{n+3}X_2$ and hence a homology Bockstein $H_{n+3}X_2 \longrightarrow H_{n+2}X_2$ which is onto, though no longer one-to-one. This gives $\pi_{n+2}S^n = \pi_{n+2}X_1 = \pi_{n+2}X_2 = \mathbb{Z}/2$.

Thus far we have found that $\pi_{n+1}S^n = \pi_{n+2}S^n = \mathbb{Z}/2$. To compute $\pi_{n+3}S^n$ by these methods requires considerably more work, since the 2-primary part of the group turns out to be $\mathbb{Z}/8$, and therefore involves the tertiary operation β_3. To see this carried out quite effectively, see the book by Mosher and Tangora [**32**] which has quite unfortuately been allowed to go out of print.

0.3. Adams' innovation

Though we have made some progress, it is clear that the further we go above the connectivity of S^n, the harder the task will become. Adams' brilliant innovation was quite simple: rather than kill only the bottom homotopy group at each step, kill *all* the homotopy which can be detected in (co)homology. That is, consider a map $X \longrightarrow K$ which induces an epimorphism in cohomology, where K is an appropriate product of $K(\mathbb{Z}/p, n)$'s. In the stable range, the cohomology of the fiber will simply be the kernel of the map $H^*K \longrightarrow H^*X$. Also, in the stable range, H^*K is a free module over the Steenrod algebra. Then, by repeating this process, we will have constructed an inverse sequence of spaces whose long exact cohomology sequences will form a free resolution of H^*X over the Steenrod algebra in the stable range. The resulting inverse sequence gives rise to the spectral sequence described in the introduction.

Even if we approach them in the most unsophisticated way, the calculations are less involved. For example, suppose we wish to reproduce the fact that $\pi_{n+1}S^n = \pi_{n+2}S^n = \mathbb{Z}/2$. Let us work stably, so that we may take $n = 0$. We start with $S \longrightarrow H\mathbb{Z}$ as before, but now take Y_1 to be the cofiber so that $S \longrightarrow H\mathbb{Z} \longrightarrow Y_1$ induces a short exact sequence of $\mathcal{A}$-modules. At the next step we take Y_2 to be the cofiber of the map $Y_1 \longrightarrow K(\mathbb{Z}/2, 2) \vee K(\mathbb{Z}/2, 4)$ which sends the fundamental classes ι_2 and ι_4 to Sq^2 and Sq^4. This is onto in dimensions 5 and below. The kernel, the cohomology of Y_2, in these dimensions is then

$$H^iY_2 = \begin{cases} 0 & i < 4 \\ <Sq^2\iota_2> & i = 4 \\ <Sq^3\iota_2> \oplus <Sq^1\iota_4 + Sq^2Sq^1\iota_2> & i = 5 \end{cases}$$

The next step in the resolution is the map $Y_2 \longrightarrow K(\mathbb{Z}/2, 4) \vee K(\mathbb{Z}/2, 5)$ which sends the fundamental classes to $Sq^2\iota_2$ and $Sq^1\iota_4 + Sq^2Sq^1\iota_2$. The sequence

$$0 \longleftarrow H^*S \longleftarrow H^*H\mathbb{Z} \longleftarrow H^*(K(\mathbb{Z}/2, 2) \vee K(\mathbb{Z}/2, 4)) \longleftarrow H^*(K(\mathbb{Z}/2, 4) \vee K(\mathbb{Z}/2, 5))$$

is exact in dimensions less than 6, and provides columns 0, 1, and 2 of the Adams spectral sequence for the sphere (Figure 1.2), as well as the bottom two nonzero groups in column 3. This is sufficient to show that $\pi_0S = \mathbb{Z}$, that $\pi_1S = \pi_2S = \mathbb{Z}/2$, and that π_3S contains a $\mathbb{Z}/4$.

More significant than this computational efficiency is the fact that this reformulation of the problem allows us to bring to bear the homological algebra of modules over the Steenrod algebra, with profound consequences.

Adams' vanishing and periodicity theorems [**3**] are theorems in homological algebra. They allowed a complete analysis of the Image of the J-homomorphism [**4, 24**], and were precursors to the nilpotence and periodicity theorems of Devinatz, Hopkins, and Smith [**16**].

Those nilpotence theorems used the Adams spectral sequence as an essential tool as follows. Consider the Adams spectral sequence converging to homotopy groups we are interested in. If we can show that the Adams spectral sequence vanishes above some line, and if powers of an element will always pass this line eventually, as they must if the element has high enough filtration, then the element must be nilpotent.

The Adams spectral sequence also provides a very clean and calculationally efficient way to organize the information contained in primary, secondary, tertiary, and higher cohomology operations. This is explained in Adams' paper "On the Non-existence of Elements of Hopf Invariant One" [**2**], a paper which every student of algebraic topology should read.

CHAPTER 1

The Adams Spectral Sequence

We set up the Adams spectral sequence and identify its E_2 and E_∞ terms. We then give a brief introduction to the Steenrod algebra, some of its subalgebras, and to the cohomology of theories we shall later consider.

1.1. Adams resolutions

Let us write H for $H\mathbb{F}_p$ and $\mathcal{A} = H^*H$ for the mod p Steenrod algebra. We will work with bounded below spectra of finite type. Of course, this means that whenever we make a construction, it will either be evident that the spectra we construct have these properties, or we shall have to demonstrate that they do. The reason we want these assumptions is that we want each cofiber in an Adams resolution to have two properties which will not otherwise hold simultaneously.

First, its mod p cohomology should be a free module over the Steenrod algebra. This will hold if it is a wedge (coproduct) of suspensions of H. Second, maps into it should be determined by their effect in mod p cohomology, and this will hold if it is a *product* of suspensions of H. Under the finite type and bounded below hypothesis, the natural map from the coproduct into the product is a homotopy equivalence, and we can have both these properties at once.

Definition 1.1.1. An (ordinary mod p) *Adams resolution* of Y is an inverse sequence

$$Y \;\simeq\; Y_0 \xleftarrow{i_0} Y_1 \xleftarrow{i_1} Y_2 \xleftarrow{i_2} \cdots$$

such that for each $s = 0, 1, 2, \dots$

(1) the cofiber Ci_s is a wedge of (suspensions of) $H\mathbb{F}_p$'s, and
(2) the natural map $H^*Ci_s \longrightarrow H^*Y_s$ is an epimorphism.

Lemma 1.1.2. *Adams resolutions exist.*

Proof: We use induction on s. Since Y_s has finite type, the cohomology H^*Y_s can be generated as an $\mathcal{A}$-module by a set of elements $\{x_\alpha \in H^{n_\alpha}Y_s\}$ which is finite in each degree. It follows that the natural map

$$\bigvee_\alpha \Sigma^{n_\alpha} H\mathbb{F}_p \longrightarrow \prod_\alpha \Sigma^{n_\alpha} H\mathbb{F}_p$$

is an equivalence, so the map $Y_s \longrightarrow \prod_\alpha \Sigma^{n_\alpha} H\mathbb{F}_p$ representing $\{x_\alpha \in H^{n_\alpha}Y_s\}$ can be lifted to $\bigvee_\alpha \Sigma^{n_\alpha} H\mathbb{F}_p$. Let $Y_{s+1} \longrightarrow Y_s$ be the fiber of this lift. Then (1) and (2) are satisfied by construction and Y_{s+1} has finite type, so that the induction can continue. □

Lemma 1.1.3. *For any Adams resolution, the sequence of natural maps*

$$0 \longleftarrow H^*Y \longleftarrow H^*Ci_0 \longleftarrow H^*\Sigma Ci_1 \longleftarrow H^*\Sigma^2 Ci_2 \longleftarrow \cdots$$

is a resolution of H^*Y by free $\mathcal{A}$-modules.

Proof: Each cofiber sequence

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{p_s} Ci_s \xrightarrow{\delta_s} \Sigma Y_{s+1}$$

induces a short exact sequence in cohomology

$$0 \longleftarrow H^*Y_s \xleftarrow{p_s^*} H^*Ci_s \xleftarrow{\delta_s^*} H^*\Sigma Y_{s+1} \longleftarrow 0$$

by condition (2). Suspend them so they may be spliced to form a long exact sequence:

$$\begin{array}{ccccccccc}
0 \longleftarrow H^*Y \longleftarrow & H^*Ci_0 & \longleftarrow\!\!\longleftarrow\!\!\longleftarrow & & H^*\Sigma Ci_1 & \longleftarrow\!\!\longleftarrow\!\!\longleftarrow & & \cdots \\
& & \nwarrow & \swarrow & & \nwarrow & \swarrow & \\
& & & H^*\Sigma Y_1 & & & H^*\Sigma^2 Y_2 &
\end{array}$$

By (1), each $H^*\Sigma^s Ci_s$ is a free $\mathcal{A}$-module. $\square$

1.2. The comparison theorem

As usual in homological algebra, we have a *Comparison Theorem.*

Theorem 1.2.1. *Given Adams resolutions of Y and Z and a map $f : Y \longrightarrow Z$, there is a map $\{f_i : Y_i \longrightarrow Z_i\}$ of resolutions extending f.*

$$\begin{array}{ccccccc}
Y & \xleftarrow{i_0} & Y_1 & \xleftarrow{i_1} & Y_2 & \xleftarrow{i_2} & \cdots \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
Z & \xleftarrow{j_0} & Z_1 & \xleftarrow{j_1} & Z_2 & \xleftarrow{j_2} & \cdots
\end{array}$$

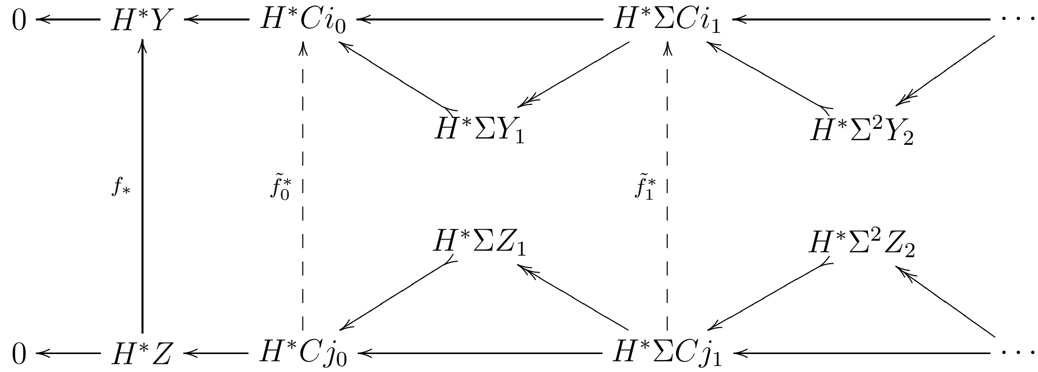
That is, $f_0 = f$ and $j_{s-1}f_s \simeq f_{s-1}i_s$ for each s. If $\{f_i\}$ and $\{\bar{f}_i\}$ are two such, then they induce chain homotopic maps of the associated algebraic resolutions.

Proof: There are two key facts. First, Cj_s is a coproduct of $H\mathbb{F}_p$'s, and hence H^*Cj_s is a free $\mathcal{A}$-module. Second, being locally finite, Cj_s is also the product of the same factors, so that a map into Cj_s is determined by the images of the fundamental classes of the factors and hence by its induced map in cohomology. Thus, for any X,

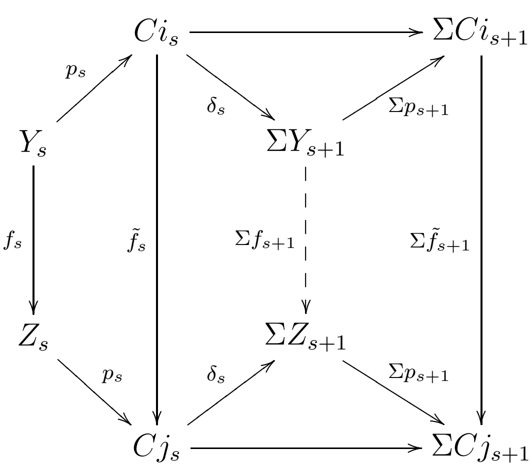
$$[X, Cj_s] \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(H^*Cj_s, H^*X). \tag{1}$$

We will use this repeatedly to show maps into Cj_s are equal by showing that their induced homomorphisms in cohomology are equal, and to construct maps by constructing their induced homomorphisms.

We are given



The comparison theorem of homological algebra asserts the existence of homomorphisms $\tilde{f}_s^*$ lifting f^* and unique up to chain homotopy. The isomorphism (1) shows they are induced by maps $\tilde{f}_s : Ci_s \longrightarrow Cj_s$. Further, (1) also tells us that the right hand outer square in the following diagram commutes, because it does so in cohomology.



We may assume inductively that f_s exists making the left square commute in cohomology, since we are given this for $f_0 = f$. But this square maps into Cj_s, so that (1) shows it also commutes up to homotopy. Thus we have an induced map $\Sigma\tilde{f}_{s+1}$ on the cofiber of the map p_s making the square containing the map δ_s commute: $\Sigma\tilde{f}_{s+1}\delta_s = \delta_s\tilde{f}_s$. The inductive step is completed by noting that the inner square containing the maps p_{s+1} must commute by (1) if it does so in cohomology, which it does since it does so after composing with the monomorphism δ_s^*. $\square$

1.3. The Adams spectral sequence

Definition 1.3.1. The (mod p cohomology) *Adams spectral sequence* for $[X, Y]$ is the spectral sequence of the exact couple

$$\begin{array}{ccc} \bigoplus_s [X, Y_s]_* & \xrightarrow{\quad i \quad} & \bigoplus_s [X, Y_s]_* \\ & {}_{\delta}\nwarrow \quad \swarrow_{p} & \\ & \bigoplus_s [X, Ci_s]_* & \end{array}$$

graded so that

$$E_1^{s,t} = [X, Ci_s]_{t-s} = [\Sigma^t X, \Sigma^s Ci_s]$$

and

$$D_1^{s,t} = [X, Y_s]_{t-s} = [\Sigma^t X, \Sigma^s Y_s].$$

The map $i = \oplus_s i_{s*}$ has bidegree $(s, t) = (-1, -1)$, $p = \oplus_s p_{s*}$ has bidegree $(0, 0)$, and $\delta = \oplus_s \delta_{s*}$ has bidegree $(1, 0)$. An element of $E_r^{s,t}$ is said to have *filtration* s, *total degree* $t - s$, and *internal degree* t.

It is traditional and convenient to display the Adams spectral sequence with the geometrically significant total degree $t - s$ increasing horizontally to the right and the filtration s increasing vertically (see Figure 1.2). The differential d_r then maps left 1 and up r.

Theorem 1.3.2. *The Adams spectral sequence for* $[X, Y]$ *has*

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*Y, H^*X)$$

and $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$. *The spectral sequence is natural in both* X *and* Y. *If* Y *is bounded below then the spectral sequence converges to* $[X, Y^{\wedge}_{(p)}]$ *where* $Y^{\wedge}_{(p)}$ *is the* p*-completion of* Y. *That is, the groups* $E_\infty^{s,t}$ *are the filtration quotients of a complete filtration of* $[X, Y^{\wedge}_{(p)}]_{t-s}$.

Remark 1.3.3.

(1) $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is graded as follows: elements of $\mathrm{Hom}^t_{\mathcal{A}}(M, N)$ lower codegrees by t. Thus, if $f \in [X, Y]_t = [\Sigma^t X, Y]$ then $f^* \in \mathrm{Hom}^t_{\mathcal{A}}(H^*Y, H^*X)$.

(2) The edge homomorphism

$$[X, Y]_* \longrightarrow \mathrm{Ext}^0(H^*Y, H^*X) = \mathrm{Hom}(H^*Y, H^*X)$$

sends a map to the homomorphism it induces in cohomology, as will be evident from the proof of the theorem. When $X = S$, this is simply the Hurewicz homomorphism in mild disguise:

$$\pi_* Y = [S, Y]_* \longrightarrow \mathrm{Hom}(H^*Y, \mathbb{F}_p) \;\cong\; H_* Y.$$

(3) We will generally work in a p-complete setting and omit notation for the p-completion, as we have just done in the preceding remarks.

Proof: As we have already observed in the proof of the Comparison Theorem 1.2.1,

$$[X, Ci_s] \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(H^*Ci_s, H^*X).$$

Similarly, the boundary map of the exact couple, the composite

$$\begin{array}{ccccc} & & [X, Y_{s+1}]_* & & \\ & \stackrel{p}{\swarrow} & & \stackrel{\delta}{\nwarrow} & \\ [X, Ci_{s+1}]_* & & & & [X, Ci_s]_* \end{array}$$

is exactly the map $\mathrm{Hom}_{\mathcal{A}}(H^*\Sigma^{s+1}Ci_{s+1}, H^*X) \longleftarrow \mathrm{Hom}_{\mathcal{A}}(H^*\Sigma^s Ci_s, H^*X)$ produced by applying $\mathrm{Hom}_{\mathcal{A}}(\bullet, H^*X)$ to the free resolution associated to the Adams resolution $\{Y_s\}$. Therefore, E_2 is $\mathrm{Ext}_{\mathcal{A}}(H^*Y, H^*X)$ by Lemma 1.1.3. This proves the first statement in the theorem.

The differential d_r is the composite of δ, $r-1$ applications of i^{-1}, and p:

$$\begin{array}{ccccc} & [X, Y_{s+r}]_{t-s-1} \xrightarrow{i} \cdots \xrightarrow{i} [X, Y_{s+1}]_{t-s-1} & \\ \stackrel{p}{\swarrow} & & \stackrel{\delta}{\nwarrow} \\ [X, Ci_{s+r}]_{t-s-1} & & [X, Ci_s]_{t-s} \\ \| & & \| \\ E_1^{s+r,t+r-1} & & E_1^{s,t} \end{array}$$

proving the second statement in the theorem.

Naturality in X is evident. Naturality in Y follows from the Comparison Theorem 1.2.1.

For the convergence of the spectral sequence, let

$$Y_\omega = \varprojlim Y_s = \mathrm{Fiber}(\prod Y_s \stackrel{\prod(1-i_s)}{\longrightarrow} \prod Y_s)$$

and consider the natural maps

$$\begin{array}{ccccccc} Y & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Y/Y_\omega & \longleftarrow & Y_1/Y_\omega & \longleftarrow & Y_2/Y_\omega & \longleftarrow & \cdots \end{array}$$

The map $\{Y_s\} \longrightarrow \{Y_s/Y_\omega\}$ induces an isomorphism of E_1 terms, and hence of the whole spectral sequence, since the two sequences have the same cofibers,

$$\mathrm{Cofiber}(Y_{s+1} \longrightarrow Y_s) \ \simeq \ \mathrm{Cofiber}(Y_{s+1}/Y_\omega \longrightarrow Y_s/Y_\omega).$$

Thus, it suffices to show

(1) the second spectral sequence is conditionally convergent to $[X, Y/Y_\omega]$, and
(2) $Y_\omega \ \simeq \ *$ when Y is p-complete.

For (1) we need

$$\varprojlim [X, Y/Y_\omega] = 0$$

and

$$\varprojlim{}^1 [X, Y/Y_\omega] = 0.$$

But the Milnor $\varprojlim{}^1$ exact sequence is

$$0 \longrightarrow \varprojlim{}^1[\Sigma X, Y_s/Y_\omega] \longrightarrow [X, \varprojlim(Y_s/Y_\omega)] \longrightarrow \varprojlim[X, Y_s/Y_\omega] \longrightarrow 0$$

and since $\varprojlim$ preserves cofibrations,

$$\varprojlim(Y_s/Y_\omega) \simeq (\varprojlim Y_s)/Y_\omega = Y_\omega/Y_\omega \simeq *$$

showing the middle term is 0, so the outer two must be as well. For (2), we simply need to show $H^*Y_\omega = 0$ since Y_ω will be p-complete when Y is. In the defining cofiber sequence

$$Y_\omega \longrightarrow \prod Y_s \stackrel{\prod(1-i_s)}{\longrightarrow} \prod Y_s$$

we may replace the products by coproducts since each Y_s is locally finite and bounded below and the connectivity of Y_s increases with s by the definition of an Adams resolution. Now the fact that $i_s^* = 0$ in H^* shows that $\bigvee(1-i_s)$ is an equivalence, and we are done. (NOT SO FAST: THIS IS FALSE FOR S or IT'S p-COMPLETION: MORE IS NEEDED. FOR FINITE Y IT IS TRUE AND NOW p-COMPLETE Y ARE LIMITS OF FINITE Y. HERE FINITE MEANS FINITE COHOMOLOGY OVER THE p-ADICS IN EACH DEGREE. CHECK THAT THIS WORKS.) □

For various constructions in the the Adams spectral sequence it will be helpful to have a geometric characterization of the terms of the spectral sequence. Let us write

$$Y_{s,r} = \text{Cofiber}(Y_{s+r} \longrightarrow Y_s)$$

so that we have cofiber sequences

$$Y_{s+r} \longrightarrow Y_s \longrightarrow Y_{s,r} \longrightarrow \Sigma Y_{s+r}$$

and

$$Y_{s+r,p} \longrightarrow Y_{s,r+p} \longrightarrow Y_{s,r} \longrightarrow \Sigma Y_{s+r,p}.$$

THEOREM 1.3.4. *The maps* $Y_{s,r} \longrightarrow Y_{s,1}$ *and* $Y_{s-r+1,r-1} \longrightarrow \Sigma Y_{s,1}$ *in the preceding cofiber sequences induce isomorphisms*

$$\begin{aligned} E_r^{s,t} &= \frac{\text{Im}([X, Y_{s,r}]_{t-s} \longrightarrow [X, Y_{s,1}]_{t-s})}{\text{Im}([X, Y_{s-r+1,r-1}]_{t-s+1} \longrightarrow [X, Y_{s,1}]_{t-s})} \\ &= \frac{\text{Im}([X, Y_{s,r}]_{t-s} \longrightarrow [X, Y_{s,1}]_{t-s})}{\text{Ker}([X, Y_{s,1}]_{t-s} \longrightarrow [X, Y_{s-r+1,r}]_{t-s})} \end{aligned}$$

The differential d_r *is induced by lifting to* $Y_{s,r}$ *and composing with the composite* $Y_{s,r} \longrightarrow \Sigma Y_{s+r} \longrightarrow Y_{s+r,1}$.

Proof: These are generic descriptions of the E_r term of the spectral sequence obtained by applying $[X, \bullet]$ to an inverse sequence. □

We will find the following characterization of Adams filtration useful.

THEOREM 1.3.5. *A map* $f \in [X, Y]$ *has Adams filtration* s *iff it factors as the composite of* s *maps which are 0 in cohomology.*

Proof: Suppose f has Adams filtration s. Then f lifts to Y_s, where

$$Y \longleftarrow Y_1 \longleftarrow \cdots \longleftarrow Y_s \longleftarrow \cdots$$

is an Adams resolution. Thus, f is the composite of the $s-1$ maps $Y \longleftarrow Y_1 \cdots \longleftarrow Y_{s-1}$ and the map $Y_{s-1} \longleftarrow Y_s \longleftarrow X$.

Conversely, suppose f factors as $f_s \cdots f_2 f_1$ with each $f_i : X_i \longrightarrow X_{i-1}$ inducing 0 in cohomology. Then by induction, there are maps $\tilde{f}_i : X_i \longrightarrow Y_i$ making the following diagram commute

$$\begin{array}{ccccccccc}
Y & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 & \longleftarrow & \cdots & \xleftarrow{f_s} & X \\
\| & & \downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle \tilde{f}_2} & & & & \downarrow{\scriptstyle \tilde{f}_s} \\
Y & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots & \longleftarrow & Y_s
\end{array}$$

(where $\tilde{f}_0 = 1_Y$) since maps into Y_i/Y_{i+1} are determined by their induced map in cohomology, and each $f_i^* = 0$ in cohomology. □

1.4. The Milnor basis for the Steenrod algebra

So that the examples are easy to check and to follow, here is a quick introduction to the Steenrod algebra expressed in terms of the Milnor basis. For proofs and more complete results, Milnor's original paper [**27**] is an excellent source.

Milnor analyzed the mod 2 Steenrod algebra as follows. Since the indecomposables of $\mathcal{A}_*$ are dual to the primitives of $\mathcal{A}$, the first task is to find these. He showed they are the Q_i, $i \geq 0$, given by

$$Q_0 = Sq^1 \tag{2}$$

$$Q_i = [Sq^{2^i}, Q_{i-1}] \quad \text{if } i > 0 \tag{3}$$

This means that $\mathcal{A}_*$ is generated by the duals ξ_{i+1} of the Q_i. The remarkable fact is that they freely generate: $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$. Now it suffices to determine the coproducts on the ξ_i, and Milnor showed they have the simple form

$$\psi(\xi_r) = \sum_{i=0}^{r} \xi_{r-i}^{2^i} \otimes \xi_i$$

Now we can dualize back: the monomials $\xi_1^{r_1} \ldots \xi_k^{r_k}$ are a basis for $\mathcal{A}_*$, and so their duals $Sq(R)$, $R = (r_1, \ldots, r_k)$, form a basis for $\mathcal{A}$. Note that the degree of $Sq(R)$ is $\Sigma(2^i - 1)r_i$.

It is not hard to check that if $R = (r)$, i.e., $(r, 0, 0, \ldots)$, then $Sq(R) = Sq^r$, so we sometimes write Sq^R or $Sq^{r_1,\ldots,r_k}$ for $Sq(R)$ for consistency of appearance. In charts of resolutions or other highly detailed contexts, we will abbreviate $Sq(R)$ to (R) to conserve space. *Note that this is* not *the same as the composite* $Sq^{r_1} Sq^{r_2} \cdots Sq^{r_k}$.

The coproduct on $\mathcal{A}$ is quite simple:

$$\psi(Sq(R)) = \sum_{R_1 + R_2 = R} Sq(R_1) \otimes Sq(R_2)$$

where sequences are added termwise, since this is dual to the multiplication of monomials:

$$\xi^{R_1} \xi^{R_2} = \xi^{R_1 + R_2}.$$

To describe the product on $\mathcal{A}$, we need to determine all monomials ξ^T whose coproduct can contain a term $\xi^R \otimes \xi^S$. This can be done in closed form, in contrast to the need to apply the Adem relations recursively when using the admissable basis. The formula is

$$Sq(R)Sq(S) = \sum_{\substack{R(X)=R \\ S(X)=S}} b(X)T(X)$$

summed over all matrices of nonnegative integers

$$X = \begin{pmatrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{4}$$

with $(0,0)$ entry omitted, whose row sum $R(X) = R$ and column sum $S(X) = S$, where

$$R(X)_r = \Sigma 2^i x_{ri} \tag{5}$$

$$S(X)_r = \Sigma x_{ir}. \tag{6}$$

The coefficient

$$b(X) = \prod_r (x_{r0}, x_{r-1,1}, \ldots, x_{0r}) \pmod 2$$

is the product of multinomial coefficients, and the term

$$T(X)_r = \sum x_{i,r-i}.$$

The computation is enormously simplified by the fact that the multinomial coefficient $(a_1, a_2, \ldots, a_k) \equiv 0 \pmod 2$ if and only if some power of 2 occurs in the base 2 expansions of at least two of the integers a_i. If you have not seen this fact, it follows by rewriting

$$(a_1, \ldots, a_k) = (a_1, a_2)(a_1 + a_2, a_3)(a_1 + a_2 + a_3, a_4) \cdots (a_1 + \cdots + a_{k-1}, a_k)$$

so that we need consider only binomial coefficients (a, b). Now, in the polynomial ring $\mathbb{F}_2[x, y]$ consider the coefficient of the term $x^a y^b$ in $(x + y)^{(a+b)}$. Writing $a = \Sigma a_i 2^i$ and $b = \Sigma b_i 2^i$ and using the fact that $(x + y)^{2^i} = x^{2^i} + y^{2^i}$, we see that $(a, b) \equiv \prod (a_i, b_i) \pmod 2$. This leaves only four cases to consider: $(0, 0) = (0, 1) = (1, 0) = 1$, while $(1, 1) = 2!/1!1! = 2 \equiv 0 \pmod 2$. If we replace 2 by p, the same argument goes through to show that (a, b) is nonzero $\pmod p$ iff each $a_i + b_i < p$.

When performing the multiplication, there is a natural order in which to generate the matrices X, namely reverse lexicographic in the rows, where the rows are themselves ordered reverse lexicographically. For example, to compute $Sq(4, 2)Sq(2, 1)$, the possible entries for row 1 are ordered as

$$(4, 0, 0) < (2, 1, 0) < (0, 2, 0) < (0, 0, 1)$$

and those for row 2 as

$$(2, 0, 0) < (0, 1, 0)$$

The matrices and their contributions to the product are then, in order,

(1) $\begin{array}{c|cc} & 2 & 1 \\ \hline 4 & 0 & 0 \\ 2 & 0 & 0 \end{array}$ $\quad (4,2)(2,0,1)\ Sq(6,3) = Sq(6,3)$

(2) $\begin{array}{c|cc} & 1 & 1 \\ \hline 2 & 1 & 0 \\ 2 & 0 & 0 \end{array}$ $\quad (2,1)(2,1,1)\ Sq(3,4) = 0$

(3) $\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 2 & 0 \\ 2 & 0 & 0 \end{array}$ $\quad (0,0)(2,2,1)\ Sq(0,5) = 0$

(4) $\begin{array}{c|cc} & 2 & 0 \\ \hline 0 & 0 & 1 \\ 2 & 0 & 0 \end{array}$ $\quad (0,2)(2,0,0)(0,1)\ Sq(2,2,1) = Sq(2,2,1)$

(5) $\begin{array}{c|cc} & 1 & 1 \\ \hline 4 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ $\quad (4,1)(0,0,1)(1,0)\ Sq(5,1,1) = Sq(5,1,1)$

(6) $\begin{array}{c|cc} & 0 & 1 \\ \hline 2 & 1 & 0 \\ 0 & 1 & 0 \end{array}$ $\quad (2,0)(0,1,1)(1,0)\ Sq(2,2,1) = 0$

(7) $\begin{array}{c|cc} & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}$ $\quad (0,1)(0,0,0)(1,1)\ Sq(1,0,2) = 0$

The multinomial coefficients (2,1,1), (2,2,1), (0,1,1) and (0,1,1,0) in the second, third, sixth and seventh matrices are zero mod 2, since the entries in (2,1,1), (0,1,1) and (0,1,1,0)=(1,1) share a 1 bit, while the entries in (2,2,1) share a 2 bit. Thus the first, fourth and fifth are the terms with nonzero coefficients, and we find that

$$Sq(4,2)Sq(2,1) = Sq(6,3) + Sq(2,2,1) + Sq(5,1,1).$$

One virtue of this ordering for hand calculation is that you start by writing R down the left edge and S along the top, and then proceed to distribute each r_i across its row, limited by the amount available at the top. When a row is distributed as far as possible to the right, at the next step you return it to its starting state, $(r_i, 0, \ldots)$ and advance the next row one step.

1.5. A minimal resolution of the Steenrod algebra in low degrees

Figure 1.1 is a minimal resolution of $\mathbb{F}_2$ over $\mathcal{A}$ through $t-s = 13$. Since the resolution is minimal, its free generators are dual to a basis for $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, which is displayed in Figure

1.2 with the duality given in Figure 1.3. In the chart (Figure 1.2) each dot represents an $\mathbb{F}_2$ summand and the three types of lines represent multiplication by h_0, h_1 and h_2.

The resolution was produced by the computer programs described in Chapter 10. There are many other ways of computing $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ of course. An explicit resolution has the advantage of carrying a lot of additional information, such as products and Massey products. In this range the Adams spectral sequence for the mod 2 homotopy groups of spheres has no differentials so that the chart in Figure 1.2 gives nearly complete information on the 2-primary part of the homotopy groups of spheres through dimension 13.

Several relations can be read off from the resolution, as will be explained carefully in Section 2.4. Briefly, suppose a and b are cocycles dual to elements α and β of the resolution, respectively: $a(\alpha) = 1$ and $b(\beta) = 1$. If $d(\alpha)$ contains $Sq^{2^i}(\beta)$, then, when we express $h_i b$ in terms of a basis containing a, the coefficient of a will be 1. For example, h_0^2 is dual to η_{00} since h_0 is dual to η_0 and $d(\eta_{00})$ contains the term $Sq^1(\eta_0)$. Thus, the relation $h_2(h_0^2) = h_1(h_1^2) = h_0(h_0 h_2)$ follows from the differential

$$d(\eta_{111}) = Sq^4\eta_{00} + Sq^2\eta_{11} + Sq^1\eta_{02}$$

in bidegree $(s,t) = (3,6)$: each of these three products is dual to η_{111} by the prescription above. This is reflected in the chart (Figure 1.2) by the fact that the diagonal, vertical and dashed lines meet at $s = 3$, $t - s = 3$. Similarly, but not visible in the chart, $h_2^3 = h_1^2 h_3$, the $i = 1$ instance of a general relation $h_{i+1}^3 = h_i^2 h_{i+2}$ which we shall deduce from the $i = 0$ case in Chapter 4. Figure 1.2 also displays the relation $h_1^2 P^1 h_1 = h_0^2 P^1 h_2$, which is deduced in the same way.

1.6. A hierarchy of homology theories and operations

We also obtain very nice descriptions of the homology and cohomology of some useful spectra in terms of the Milnor basis.

Let us define subalgebras of the mod 2 Steenrod algebra $\mathcal{A}$,

$$\begin{aligned} E(n) &= E[Q_0, \ldots, Q_n], \text{ and} \\ \mathcal{A}(n) &= \langle Sq^1, \ldots, Sq^{2^n} \rangle, \end{aligned}$$

the exterior algebra generated by Q_0 through Q_n, and the subalgebra generated by Sq^{2^i}, $i \leq n$, respectively. These are finite sub Hopf algebras of $\mathcal{A}$ with duals

$$\begin{aligned} E(n)_* &= E[\bar{\xi}_1, \ldots, \bar{\xi}_{n+1}] \text{ and} \\ \mathcal{A}(n)_* &= \mathcal{A}_*/(\bar{\xi}_1^{2^{n+1}}, \bar{\xi}_2^{2^n}, \ldots, \bar{\xi}_{n+1}^2, \bar{\xi}_{n+2}, \ldots) \end{aligned}$$

where $\bar{\xi}_i$ is the conjugate of ξ_i. Note that $\mathcal{A}_*$ could equally well be described as the polynomial algebra on the $\bar{\xi}_i$ with coproduct

$$\psi(\bar{\xi}_r) = \sum_{i=0}^{r} \bar{\xi}_i \otimes \bar{\xi}_{r-i}^{2^i}$$

One virtue of the dual description of $\mathcal{A}(n)$ is that it is easy to see that its dimension over $\mathbb{F}_2$ must be $2^{n+1}2^n \cdots 2 = 2^{(n+1)(n+2)/2}$, and that its top nonzero degree must be

$$\begin{aligned}(2^{n+1}-1)+(2^n-1)+\cdots+(2-1) &= \sum_{i=1}^{n+1}(2^{n+2-i}-1)(2^i-1) \\ &= (n+1)(2^{n+2}+1)-2^{n+3}+4\end{aligned}$$

the degree in which $\xi_1^{2^{n+1}-1}\xi_2^{2^n-1}\cdots\xi_{n+1}$ occurs.

A number of important spectra have mod 2 cohomology which is simply described in terms of these subalgebras. We will see in Chapter 8 that this can make calculation of the associated cohomology theories realtively accessible.

Let $BP{<}n{>}$ be obtained from the Brown-Peterson spectrum BP by killing v_{n+1}, v_{n+2}, etcetera. Then $BP{<}0{>}= H\mathbb{Z}_{(p)}$, and $BP{<}1{>}$ is the Adams summand of the p-localization of complex connective K-theory, ku, so that

$$ku_{(p)} \simeq BP{<}1{>} \vee \Sigma^2 BP{<}1{>} \vee \cdots \vee \Sigma^{2(p-2)}BP{<}1{>} .$$

Let ko be real connective K-theory and eo_2 the connective elliptic cohomology theory constructed by Hopkins and Mahowald [**18**].

Recall the notation for Hopf algebra quotients. If $\mathcal{B}$ is a sub Hopf algebra of $\mathcal{A}$ over k, then $\mathcal{A}//\mathcal{B} := \mathcal{A}\otimes_{\mathcal{B}} k$. This is the quotient of $\mathcal{A}$ by the left ideal of $\mathcal{A}$ generated by the augmentation ideal $I\mathcal{B}$ of $\mathcal{B}$. The point is just that the usual notation $\mathcal{A}/\mathcal{B}$ is bad because the left ideal generated by $\mathcal{B}$ itself is all of $\mathcal{A}$ since $1 \in \mathcal{B}$.

THEOREM 1.6.1.

$$\begin{aligned} H^*BP{<}n{>} &= \mathcal{A}//E(n) \\ H^*eo_2 &= \mathcal{A}//\mathcal{A}(2) \\ H^*ko &= \mathcal{A}//\mathcal{A}(1) \\ H^*H\mathbb{Z} &= \mathcal{A}//\mathcal{A}(0). \end{aligned}$$

Further, there are maps

$$\begin{array}{ccccccccccc} & & & & eo_2 & \longrightarrow & ko & & & & \\ & & & & \downarrow & & \downarrow & & & & \\ \cdots \longrightarrow & BP{<}n{>} & \longrightarrow \cdots \longrightarrow & & BP{<}2{>} & \longrightarrow & ku & \longrightarrow & H\mathbb{Z} & \longrightarrow & H\mathbb{F}_2 \end{array}$$

whose induced homomorphisms in cohomology are the evident quotients induced by the inclusions

$$\begin{array}{ccccccccc} & & & & \mathcal{A}(2) & \longleftarrow & \mathcal{A}(1) & & \\ & & & & \uparrow & & \uparrow & & \\ \cdots \longleftarrow & E(n) & \longleftarrow \cdots \longleftarrow & & E(2) & \longleftarrow & E(1) & \longleftarrow E(0) \longleftarrow & 1 \end{array}$$

C_0: $C_{0,0}$: $\iota_0 \mapsto \iota$

C_1: $C_{1,1}$: $\eta_0 \mapsto (1)\iota_0$
$C_{1,2}$: $\eta_1 \mapsto (2)\iota_0$
$C_{1,4}$: $\eta_2 \mapsto (4)\iota_0$
$C_{1,8}$: $\eta_3 \mapsto (8)\iota_0$

C_2: $C_{2,2}$: $\eta_{00} \mapsto (1)\eta_0$
$C_{2,4}$: $\eta_{11} \mapsto (3)\eta_0 + (2)\eta_1$
$C_{2,5}$: $\eta_{02} \mapsto (4)\eta_0 + (0,1)\eta_1 + (1)\eta_2$
$C_{2,8}$: $\eta_{22} \mapsto (7)\eta_0 + (6)\eta_1 + (4)\eta_2$
$C_{2,9}$: $\eta_{03} \mapsto ((8) + (2,2))\eta_0 + ((7) + (4,1) + (0,0,1))\eta_1 + (1)\eta_3$
$C_{2,10}$: $\eta_{13} \mapsto ((9) + (3,2))\eta_0 + ((8) + (5,1))\eta_1 + (0,2)\eta_2 + (2)\eta_3$

C_3: $C_{3,3}$: $\eta_{000} \mapsto (1)\eta_{00}$
$C_{3,6}$: $\eta_{111} \mapsto (4)\eta_{00} + (2)\eta_{11} + (1)\eta_{02}$
$C_{3,10}$: $\eta_{003} \mapsto ((8) + (2,2))\eta_{00} + ((6) + (0,2))\eta_{11} + (1)\eta_{03}$
$C_{3,11}$: $\gamma_0 \mapsto ((9) + (3,2))\eta_{00} + (0,0,1)\eta_{11} + (6)\eta_{02} + ((3) + (0,1))\eta_{22}$
$C_{3,12}$: $\eta_{222} \mapsto (10)\eta_{00} + ((8) + (1,0,1))\eta_{11} + (4)\eta_{22} + (3)\eta_{03} + (2)\eta_{13}$

C_4: $C_{4,4}$: $\eta_{0^4} \mapsto (1)\eta_{000}$
$C_{4,11}$: $\eta_{0003} \mapsto (8)\eta_{000} + ((5) + (2,1))\eta_{111} + (1)\eta_{003}$
$C_{4,13}$: $\gamma_{00} \mapsto ((10) + (4,2))\eta_{000} + ((7) + (1,2) + (0,0,1))\eta_{111} + (2)\gamma_0$

C_5: $C_{5,5}$: $\eta_{0^5} \mapsto (1)\eta_{0^4}$
$C_{5,14}$: $\rho_1 \mapsto (10)\eta_{0^4} + ((3) + (0,1))\eta_{0003}$
$C_{5,16}$: $\rho_2 \mapsto (12)\eta_{0^4} + ((5) + (2,1))\eta_{0003} + (3)\gamma_{00}$

C_6: $C_{6,6}$: $\eta_{0^6} \mapsto (1)\eta_{0^5}$
$C_{6,16}$: $\rho_{11} \mapsto (11)\eta_{0^5} + (2)\rho_1$
$C_{6,17}$: $\rho_{02} \mapsto (12)\eta_{0^5} + (0,1)\rho_1 + (1)\rho_2$

C_7: $C_{7,7}$: $\eta_{0^7} \mapsto (1)\eta_{0^6}$
$C_{7,18}$: $\rho_{111} \mapsto (12)\eta_{0^6} + (2)\rho_{11} + (1)\rho_{02}$

FIGURE 1.1. A minimal resolution of $\mathbb{F}_2$ over $\mathcal{A}$ through $t - s = 13$.

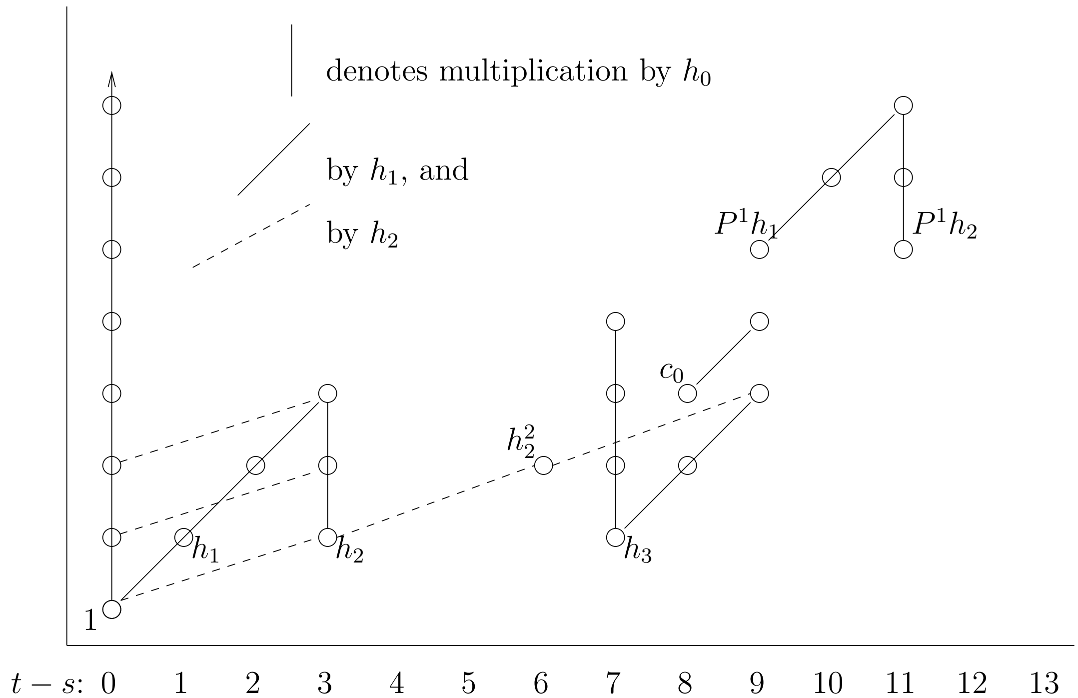


FIGURE 1.2. $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 13$

- $1 = (\iota_0)^*$.
- $h_i = (\eta_i)^*$.
- $h_j h_i = h_i h_j = (\eta_{ij})^*$ if η_{ij} or η_{ji} exist, otherwise $h_i h_j = 0$.
- Similarly, $h_i h_j h_k = (\eta_{ijk})^*$ if η_{ijk} or a permutation exist, otherwise $h_i h_j h_k = 0$, *except* $h_0^2 h_2 = h_1^3 = (\eta_{111})^*$ and $h_1^2 h_3 = h_2^3 = (\eta_{222})^*$.
- $h_0^3 h_3 = (\eta_{0003})^*$.
- $h_0^k = (\eta_{0^k})^*$ for $k > 3$.
- $c_0 = (\gamma_0)^*$.
- $h_1 c_0 = (\gamma_{00})^*$.
- $P^1 h_1 = (\rho_1)^*$.
- $P^1 h_2 = (\rho_2)^*$.
- $h_1 P^1 h_1 = (\rho_{11})^*$.
- $h_0 P^1 h_2 = (\rho_{02})^*$.
- $h_1^2 P^1 h_1 = h_0^2 P^1 h_2 = (\rho_{111})^*$.

FIGURE 1.3. Definition of cocycles generating $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 13$

CHAPTER 2

Products

There are two sorts of products to consider:

$$\begin{array}{ll} \text{composition} & [Y,Z]_p \otimes [X,Y]_q \stackrel{\circ}{\longrightarrow} [X,Z]_{p+q} \\ \text{and smash} & [X_1,Y_1]_p \otimes [X_2,Y_2]_q \stackrel{\wedge}{\longrightarrow} [X_1 \wedge X_2, Y_1 \wedge Y_2]_{p+q}. \end{array}$$

To be precise, if $f \in [Y,Z]_p$ and $g \in [X,Y]_q$ then $fg = f \circ g \in [X,Y]_{p+q}$ is the composite

$$X \wedge S^{p+q} \stackrel{1\wedge t_{q,p}}{\longrightarrow} X \wedge S^q \wedge S^p \stackrel{g\wedge 1}{\longrightarrow} Y \wedge S^p \stackrel{f}{\longrightarrow} Z$$

where $t_{q,p}$ is the one-point compactification of the evident linear isomorphism $\mathbb{R}^{p+q} \longrightarrow \mathbb{R}^q \oplus \mathbb{R}^p$. Similarly, if $f \in [X_1,Y_1]_p$ and $g \in [X_2,Y_2]_q$ then $f \wedge g \in [X_1 \wedge X_2, Y_1 \wedge Y_2]_{p+q}$ is the composite

$$X_1 \wedge X_2 \wedge S^{p+q} \stackrel{1\wedge t_{p,q}}{\longrightarrow} X_1 \wedge X_2 \wedge S^p \wedge S^q \stackrel{1\wedge\tau\wedge 1}{\longrightarrow} X_1 \wedge S^p \wedge X_2 \wedge S^q \stackrel{f\wedge g}{\longrightarrow} Y_1 \wedge Y_2.$$

(GET THESE RIGHT)

For self maps of the sphere, these products are the same: if $f \in [S,S]_p$ and $g \in [S,S]_q$ then $fg \in [S,S]_{p+q}$ is

$$S \wedge S^{p+q} \stackrel{1\wedge t_{p,q}}{\longrightarrow} S \wedge S^p \wedge S^q \stackrel{g\wedge 1}{\longrightarrow} S \wedge S^p \stackrel{1\wedge f}{\longrightarrow} S \wedge S \;\simeq\; S$$

which agrees with $g \wedge f$, while $gf \in [S,S]_{p+q}$ is

$$S^{p+q} \stackrel{t_{q,p}}{\longrightarrow} S^q \wedge S^p \stackrel{f\wedge 1}{\longrightarrow} S \wedge S^q \stackrel{1\wedge g}{\longrightarrow} S \wedge S \;\simeq\; S$$

which agrees with $f \wedge g$. These two differ by $(-1)^{pq}$, the degree of the twist map $\tau : S^p \wedge S^q \longrightarrow S^q \wedge S^p$:

$$\begin{array}{ccc} S^{p+q} & \xrightarrow{(-1)^{pq}} & S^{p+q} \\ {\scriptstyle t_{p,q}}\downarrow & & \downarrow{\scriptstyle t_{q,p}} \\ S^p \wedge S^q & \xrightarrow{\tau} & S^q \wedge S^p \end{array}$$

Both products are well behaved in the Adams spectral sequence. Let $E_r(X,Y)$ be the E_r term of the Adams spectral sequence $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*Y, H^*X) \Longrightarrow [X,Y]_{t-s}$.

2.1. Smash and tensor products

In cohomology, the smash product of maps becomes the tensor product of the corresponding homomorphisms when composed with the Künneth isomorphisms.

$$\begin{array}{ccc} H^*(Y_1 \wedge Y_2) & \xrightarrow{(f\wedge g)^*} & H^*(X_1 \wedge X_2) \\ \kappa \downarrow \cong & & \kappa \downarrow \cong \\ H^*Y_1 \otimes H^*Y_2 & \xrightarrow{f^*\otimes g^*} & H^*X_1 \otimes H^*X_2 \end{array}$$

In turn, the tensor product of homomorphisms induces pairings of Ext groups

$$\mathrm{Ext}(H^*Y_1, H^*X_1) \otimes \mathrm{Ext}(H^*Y_2, H^*X_2) \longrightarrow \mathrm{Ext}(H^*Y_1 \otimes H^*Y_2, H^*X_1 \otimes H^*X_2).$$

THEOREM 2.1.1. *There is a natural pairing of spectral sequences*

$$E_r^{s_1,t_1}(X_1, Y_1) \otimes E_r^{s_2,t_2}(X_2, Y_2) \longrightarrow E_r^{s,t}(X_1 \wedge X_2, Y_1 \wedge Y_2)$$

where $s = s_1 + s_2$ and $t = t_1 + t_2$, such that

(1) *at E_2 it is the pairing of* Ext *groups induced by the tensor product and the Künneth isomorphisms,*
(2) *d_r is a derivation with respect to this product,*
(3) *the pairing at E_{r+1} is induced from that at E_r, and*
(4) *the pairing at E_∞ is induced by the smash product.*

Proof: Smash Adams resolutions of Y_1 and Y_2 and filter the resulting bicomplex by total degree. The result is an Adams resolution of $Y_1 \wedge Y_2$ whose corresponding algebraic resolution is exactly the tensor product of the algebraic resolutions of H^*Y_1 and H^*Y_2 given by the Adams resolutions we started with. □

COROLLARY 2.1.2. *The Adams spectral sequence*

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow [S, S]^\wedge_{(p)}$$

is a spectral sequence of rings converging to the ring structure of $[S, S]$ and every Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(H^*Y, H^*X) \Longrightarrow [X, Y]^\wedge_{(p)}$$

is a module over it, with the pairing at E_∞ induced by the smash product pairing

$$[S, S] \otimes [X, Y] \longrightarrow [X, Y].$$

This means that for every r there is a pairing

$$E_r(S, S) \otimes E_r(X, Y) \longrightarrow E_r(X, Y)$$

which makes $E_r(X, Y)$ a module over the ring $E_r(S, S)$ with respect to which the differential d_r is a derivation, and that the pairing at E_{r+1} is induced by that at E_r.

2.2. Composition and Yoneda products

Composition of homomorphisms

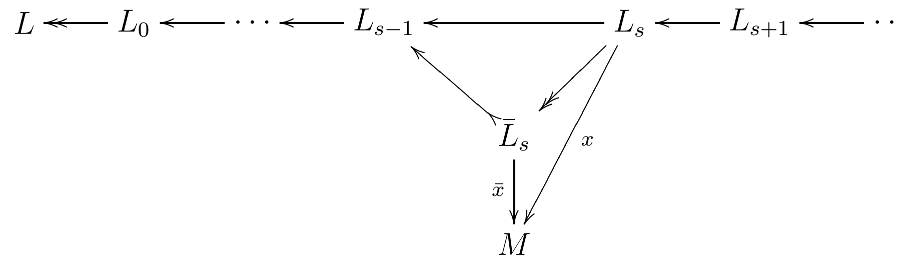
$$\mathrm{Hom}(M,N)\otimes \mathrm{Hom}(L,M) \longrightarrow \mathrm{Hom}(L,N),$$

induces a pairing of Ext groups

$$\mathrm{Ext}(M,N)\otimes \mathrm{Ext}(L,M) \longrightarrow \mathrm{Ext}(L,N),$$

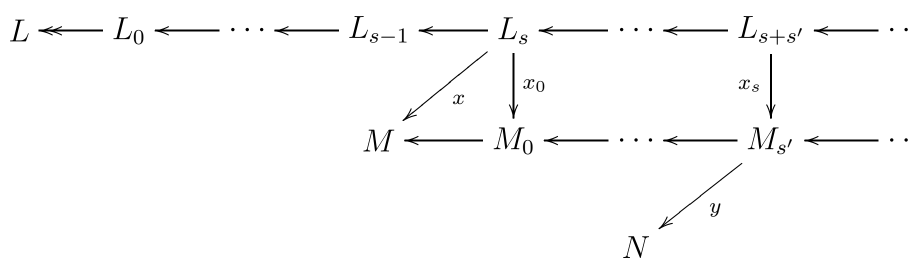
known as the *Yoneda product*. We may view elements of Ext in three related ways: as equivalence classes of cocycles, chain maps, or extensions. We shall find all three useful, so we will describe the Yoneda product in terms of each.

If $\mathcal{L} \longrightarrow L$, $\mathcal{M} \longrightarrow M$, and $\mathcal{N} \longrightarrow N$ are projective resolutions of L, M, and N, then we may define elements $x \in \mathrm{Ext}^{s,t}_{\mathcal{A}}(L,M)$ as equivalence classes of cocycles $\Sigma^t L_s \longrightarrow M$, that is, homomorphisms $\bar{x} : \Sigma^t \bar{L}_s \longrightarrow M$ modulo those which factor through L_{s-1}:



(here $\bar{L}_s = \mathrm{Cok}(L_{s+1} \longrightarrow L_s) = \mathrm{Ker}(L_{s-1} \longrightarrow L_{s-2})$). *We omit explicit mention of suspensions where possible to simplify the notation.*

In this view, the Yoneda composite $yx \in \mathrm{Ext}(L,N)$ of x and $y \in \mathrm{Ext}(M,N)$ is the (equivalence class of the) cocycle yx_s, where $\{x_s\}$ is a chain map (of degree t) lifting x:



Alternatively, elements of $\mathrm{Ext}^{s,t}(L,M)$ can be viewed as chain homotopy clases of chain maps $x_* : \mathcal{L} \longrightarrow \mathcal{M}$ of bidegree (s,t). In this view, Yoneda product is simply composition of chain maps. Though this definition is much cleaner, the definition in terms of cocyles has the virtue of making it clear that a single homomorphism, the cocycle $x : \Sigma^t L_s \longrightarrow M$, completely determines the chain equivalence class of the chain map $x_* = \{x_s\}$. To be explicit, the cocycle x is ϵx_0, where $\epsilon : M_0 \longrightarrow M$ is the augmentation of $\mathcal{M}$, and it determines the chain homotopy class of x_* by the comparison theorem.

Finally, we may view elements of $\mathrm{Ext}^{s,t}(L,M)$, for $s > 0$, as equivalence classes of extensions (exact sequences)

$$\mathcal{E}_x : \qquad 0 \longleftarrow \Sigma^t L \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots \longleftarrow P_{s-1} \longleftarrow M \longleftarrow 0,$$

and in these terms, the product yx is the sequence obtained by splicing $\Sigma^{t'}\mathcal{E}_x$ and $\mathcal{E}_y$:

$$\mathcal{E}_y: \qquad 0 \longleftarrow \Sigma^{t'}M \longleftarrow Q_0 \longleftarrow Q_1 \longleftarrow \cdots \longleftarrow Q_{s'-1} \longleftarrow N \longleftarrow 0,$$

to get

$$\begin{array}{ccccccccccccc}
0 \leftarrow \Sigma^{t+t'}L \leftarrow \Sigma^{t'}P_0 \leftarrow \cdots \leftarrow \Sigma^{t'}P_{s-1} & \longleftarrow & Q_0 \leftarrow \cdots \leftarrow Q_{s'-1} \leftarrow N \leftarrow 0 \\
\nwarrow & & \swarrow \\
& \Sigma^{t'}M &
\end{array}$$

To compare this to the preceding definitions, an extension $\mathcal{E}$ determines an equivalence class of cocycles $x_{\mathcal{E}}$ by the comparison theorem,

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & \Sigma^t L & \longleftarrow & \Sigma^t L_0 & \longleftarrow & \cdots & \longleftarrow & \Sigma^t L_{s-1} & \longleftarrow & \Sigma^t L_s & \longleftarrow & \cdots \\
 & & \| & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle x_{\mathcal{E}}} & & \\
0 & \longleftarrow & \Sigma^t L & \longleftarrow & P_0 & \longleftarrow & \cdots & \longleftarrow & P_{s-1} & \longleftarrow & M & \longleftarrow & 0
\end{array}$$

and a cocycle x defines an extension by pushout:

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & \Sigma^t L & \xleftarrow{k_0} & \Sigma^t L_0 & \xleftarrow{k_1} & \cdots & \xleftarrow{k_{s-1}} & \Sigma^t L_{s-1} & \xleftarrow{k_s} & \Sigma^t L_s & \longleftarrow & \cdots \\
 & & \| & & \downarrow{\scriptstyle x_0} & & & & \downarrow{\scriptstyle x_{s-1}} & & \downarrow{\scriptstyle x_s = x} & & \\
0 & \longleftarrow & \Sigma^t L & \xleftarrow{j_0} & P_0 & \xleftarrow{j_1} & \cdots & \xleftarrow{j_{s-1}} & P_{s-1} & \xleftarrow{j_s} & M & \longleftarrow & 0
\end{array}$$

each pair (x_{i-1}, j_i) is the pushout of the pair (k_i, x_i), with $x_s = x$.

When $x \in \mathrm{Ext}^0(L, M) = \mathrm{Hom}(L, M)$, Yoneda composite with x is simply the induced homomorphism

$$x_* : \mathrm{Ext}(K, L) \longrightarrow \mathrm{Ext}(K, M)$$

or

$$x^* : \mathrm{Ext}(M, N) \longrightarrow \mathrm{Ext}(L, N)$$

the latter of which is given by the comparison theorem. If $x = f^* : H^*Z \longrightarrow H^*Y$, the comparison theorem 1.2.1 for Adams resolutions implies the naturality of the Adams spectral sequence:

$$\begin{array}{ccc}
\mathrm{Ext}^s(H^*Y, H^*X) & \Longrightarrow & [X, Y] \\
\downarrow{\scriptstyle (f^*)^*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{Ext}^s(H^*Z, H^*X) & \Longrightarrow & [X, Z].
\end{array}$$

However, this is nontrivial in Ext only when f induces a nonzero homomorphism in cohomology, that is, when f has Adams filtration 0. The *Generalized Comparison Theorem* gives the corresponding result when f has positive Adams filtration, and is the key step in proving that composition of maps is detected by Yoneda composite at E_2.

THEOREM 2.2.1. *If the map $f : Y \longrightarrow Z$ has filtration s then f induces a map of Adams resolutions*

$$\begin{array}{ccccccccccc}
 & & & & Y & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots \\
 & \swarrow^{f} & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
Z & \longleftarrow Z_1 \longleftarrow & \cdots & \longleftarrow & Z_s & \longleftarrow & Z_{s+1} & \longleftarrow & Z_{s+2} & \longleftarrow & \cdots
\end{array}$$

Proof: By theorem 1.3.5, f lifts to $f_0 : Y \longrightarrow Z_s$. Since

$$Z_s \longleftarrow Z_{s+1} \longleftarrow \cdots$$

is an Adams resolution of Z_s, the comparison theorem 1.2.1 gives the lifts f_i for $i > 0$. □

The resulting chain map

$$\begin{array}{ccccccccc}
0 & \longleftarrow & H^*Y & \longleftarrow & H^*\bar{Y}_0 & \longleftarrow & H^*\Sigma\bar{Y}_1 & \longleftarrow & \cdots \\
 & & \uparrow{\scriptstyle f_0^*} & & \uparrow{\scriptstyle \bar{f}_0^*} & & \uparrow{\scriptstyle \bar{f}_1^*} & & \\
0 & \longleftarrow & H^*\Sigma^s Z_s & \longleftarrow & H^*\Sigma^s\bar{Z}_s & \longleftarrow & H^*\Sigma^{s+1}\bar{Z}_{s+1} & \longleftarrow & \cdots
\end{array}$$

defines a cocycle $H^*\Sigma^s\bar{Z}_s \longrightarrow H^*Y$, unique up to coboundaries, whose value in $\mathrm{Ext}^{s,s}(H^*Z, H^*Y)$ is an infinite cycle which detects $f \in [Y, Z]$.

The following theorem, the main result of this section, will imply that Yoneda composite with this cocycle will converge to the induced homomorphism $[X, Y] \longrightarrow [X, Z]$. More generally, the theorem applies at all stages of the spectral sequence, not just to infinite cycles.

THEOREM 2.2.2. *There is a natural pairing of spectral sequences*

$$E_r^{s_1,t_1}(Y, Z) \otimes E_r^{s_2,t_2}(X, Y) \longrightarrow E_r^{s,t}(X, Z)$$

where $s = s_1 + s_2$ and $t = t_1 + t_2$, such that

(1) *at E_2 it is the Yoneda pairing,*
(2) *d_r is a derivation with respect to this product,*
(3) *the pairing at E_{r+1} is induced from that at E_r, and*
(4) *the pairing at E_∞ is induced by composition of maps.*

Proof: R.M.F. Moss ([**33**]). □

2.3. The geometric boundary theorem

Suppose that $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is a cofiber sequence with geometric boundary map $\delta : C \longrightarrow \Sigma A$. If $\delta^* = 0$ in cohomology, then the cofiber sequence induces a short exact sequence

$$\mathcal{E} : \qquad 0 \longrightarrow H^*C \xrightarrow{\beta^*} H^*B \xrightarrow{\alpha^*} H^*A \longrightarrow 0$$

so there is an algebraic boundary map, or connecting homomorphism,

$$\partial : \mathrm{Ext}^{s,t}(H^*C, H^*X) \longrightarrow \mathrm{Ext}^{s+1,t}(H^*A, H^*X).$$

Theorem 2.3.1 (Bruner [**10**]). *The algebraic boundary map commutes with Adams spectral sequence differentials, and thus induces a map*

$$\partial_r : E_r^{s,t}(X, C) \longrightarrow E_r^{s+1,t}(X, A).$$

for each $r \geq 2$*. At* E_∞*,* ∂_∞ *is the map of associated graded modules induced by* $\delta : C \longrightarrow \Sigma A$*:*

$$F_s[X, C]_{t-s} \xrightarrow{\delta_*} F_{s+1}[X, \Sigma A]_{t-s} \subset [X, A]_{t-s-1}$$

where F_s *is the submodule of maps of Adams filtration at least* s*.*

Of course, if δ has Adams filtration greater than one, this last map will be trivial.

Proof: Since the algebraic boundary map ∂ is given by Yoneda composite with the extension $\mathcal{E}$, we need only show that δ is detected by $\mathcal{E}$. Since $\delta^* = 0$, δ lifts to δ_0 as in theorem 2.2.1. This induces a map of cofiber sequences:

$$\begin{array}{ccccccc}
\Sigma A & \xleftarrow{\delta} & C & \xleftarrow{\beta} & B & \xleftarrow{\alpha} & A \\
\| & & \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \bar{\delta}_0} & & \| \\
\Sigma A & \longleftarrow & \Sigma A_1 & \longleftarrow & A/A_1 & \longleftarrow & A
\end{array}$$

This in turn induces a map of short exact sequences which shows that the cocycle δ_0^* corresponds to the extension $\mathcal{E}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^*C & \longrightarrow & H^*B & \longrightarrow & H^*A & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \delta_0^*} & & \uparrow & & \| & & \\
0 & \longrightarrow & H^*\Sigma A_1 & \longrightarrow & H^*A/A_1 & \longrightarrow & H^*A & \longrightarrow & 0
\end{array}$$

By the general comparison theorem (2.2.1), we obtain a map from any Adams resolution of C to any Adams resolution of ΣA_1. Commutation of ∂ with Adams differentials and convergence to the geometric boundary map δ now follows simply by composition with this map of resolutions:

$$\begin{array}{ccccccccc}
 & & & & & & C\Sigma^{t-1}X & \longleftarrow & \Sigma^{t-1}X \\
 & & & & & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle d_r y} \\
 & & C & \longleftarrow & \cdots & \longleftarrow & C_s & \longleftarrow & C_{s+r} \\
 & \swarrow{\scriptstyle \delta} & \downarrow{\scriptstyle \delta_0} & & & & \downarrow & & \downarrow \\
\Sigma A & \longleftarrow & \Sigma A_1 & \longleftarrow & \cdots & \longleftarrow & \Sigma A_{s+1} & \longleftarrow & \Sigma A_{s+r+1}
\end{array}$$

shows commutation with the differentials, and

$$\begin{array}{ccccccc}
 & & & & & & \Sigma^t X \\
 & & & & & & \downarrow{\scriptstyle y} \\
 & & C & \longleftarrow & \cdots & \longleftarrow & C_s \\
 & \swarrow{\scriptstyle \delta} & \downarrow{\scriptstyle \delta_0} & & & & \downarrow \\
\Sigma A & \longleftarrow & \Sigma A_1 & \longleftarrow & \cdots & \longleftarrow & \Sigma A_{s+1}
\end{array}$$

shows that ∂_∞ is the map of associated graded modules given by composition with the geometric boundary map. □

Corollary 2.3.2. *The long exact sequence in Ext converges to the long exact sequence in homotopy:*

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \delta_*} \\ \mathrm{Ext}^{s,t}(H^*A, H^*X) & \Longrightarrow & [X, A]_{t-s} \\ \downarrow{\scriptstyle (\alpha^*)^*} & & \downarrow{\scriptstyle \alpha_*} \\ \mathrm{Ext}^{s,t}(H^*B, H^*X) & \Longrightarrow & [X, B]_{t-s} \\ \downarrow{\scriptstyle (\beta^*)^*} & & \downarrow{\scriptstyle \beta_*} \\ \mathrm{Ext}^{s,t}(H^*C, H^*X) & \Longrightarrow & [X, C]_{t-s} \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \delta_*} \\ \mathrm{Ext}^{s+1,t}(H^*A, H^*X) & \Longrightarrow & [X, A]_{t-s-1} \\ \downarrow{\scriptstyle (\alpha^*)^*} & & \downarrow{\scriptstyle \alpha_*} \\ \vdots & & \vdots \end{array}$$

Proof: This follows since ∂ is $(\delta_0^*)^*$ composed with the isomorphism $\mathrm{Ext}^{s,t}(H^*(\Sigma A_1), H^*X) \cong \mathrm{Ext}^{s+1,t}(H^*A, H^*X)$. □

Since $\delta^* = 0$ in cohomology, α^* is an epimorphism and β^* a monomorphism. In particular, α^* and β^* are both nonzero and the maps α_* and β_* in homotopy will not generically raise Adams filtration, whereas δ_* will raise Adams filtration by at least one. If the short exact sequence $\mathcal{E}$ is not split, then the algebraic boundary map will generically be nonzero and the map δ_* will generically raise Adams filtration by exactly one. More precisely, in the universal example, $X = C$, $\delta_*(1_C) = \delta$, so that δ_* raises Adams filtration by exactly one in this instance.

When the extension $\mathcal{E}$ is split, then $\partial = 0$, so that at E_2 the long exact sequence becomes short exact sequences. In this case, $\delta : C \longrightarrow \Sigma A$ has Adams filtration $s_0 > 1$ and Yoneda composite with the element $\delta_0 \in \mathrm{Ext}^{s_0,s_0-1}(H^*A, H^*C)$ which detects δ converges to δ_*:

$$\begin{array}{ccc} \mathrm{Ext}^{s,t}(H^*C, H^*X) & \Longrightarrow & [X, C]_{t-s} \\ \downarrow{\scriptstyle \delta_0^*} & & \downarrow{\scriptstyle \delta_*} \\ \mathrm{Ext}^{s+s_0,t+s_0-1}(H^*A, H^*X) & \Longrightarrow & [X, A]_{t-s-1} \end{array}$$

However, this no longer fits into an exact sequence at E_2.

2.4. Products by Ext^1

We can now justify our claims about products in Section 1.5. Consider products

$$\mathrm{Ext}^1(\mathbb{F}_2,\mathbb{F}_2)\otimes \mathrm{Ext}^s(H^*X,\mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s+1}(H^*X,\mathbb{F}_2).$$

We can compute these directly from the differential in the minimal free resolution of H^*X. In fact, the technique works perfectly well to compute

$$\mathrm{Ext}^1_A(k,k)\otimes \mathrm{Ext}^s_A(M,k) \longrightarrow \mathrm{Ext}^{s+1}_A(M,k)$$

for any connected algebra A over k and any module M over A.

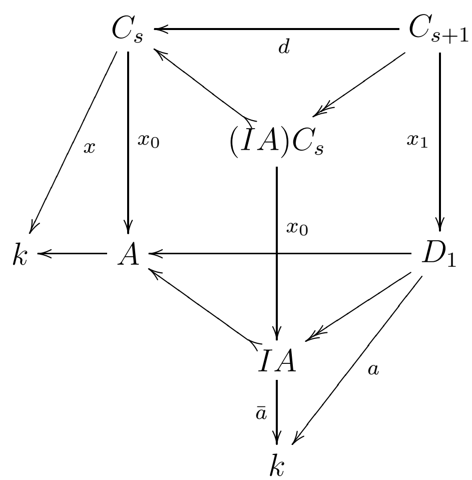
Let $IA = \mathrm{Ker}(A \xrightarrow{\epsilon} k)$ be the augmentation ideal of A. If $D_* \longrightarrow k$ is a minimal free resolution of k, then $D_0 = A$ and $d_0 : D_1 \longrightarrow D_0$ factors as $D_1 \twoheadrightarrow IA \rightarrowtail A$. It follows that $\mathrm{Ext}^1_A(k,k)$ is the linear dual of $IA/(IA)^2$, since any cocycle of homological degree one is represented by a unique linear map $IA \longrightarrow k$. (Minimality of D_* gives uniqueness.)

Let $C_* \longrightarrow M$ be a minimal free resolution of M.

PROPOSITION 2.4.1. *Let $a \in \mathrm{Ext}^1(k,k)$ and $x \in \mathrm{Ext}^s(M,k)$. Let $x_i : C_{s+i} \longrightarrow D_i$ lift the cocycle representing x, and let $\bar{a} : IA \longrightarrow k$ be the factorization of a through $IA = \mathrm{Im}(d_0)$. Then $ax_1 : C_{s+1} \longrightarrow k$ is a cocycle representing $ax \in \mathrm{Ext}^{s+1}(M,k)$. If $\{g_i\}$ is a basis for C_s and $g \in C_{s+1}$ has $d(g) = \sum \alpha_i g_i$, then the value of this cocycle can be computed directly from x by*

$$\begin{aligned}(ax_1)(g) &= \bar{a}x_0(d(g))\\ &= \bar{a}x_0(\textstyle\sum \alpha_i g_i)\\ &= \textstyle\sum \bar{a}(\alpha_i)x(g_i)\end{aligned}$$

Proof: We have already shown that the cocycle ax_1 represents ax. To show the formula for ax_1 is correct, it is only necessary to remark that the following diagram commutes and that $x_0(\alpha g_i) = \alpha x_0(g_i)$, since x_0 is determined by x and the requirement that x_0 be A-linear.



□

If $A = \mathcal{A}$, the mod 2 Steenrod algebra, then $I\mathcal{A}/(I\mathcal{A})^2$ is spanned by a minimal set of generators for $\mathcal{A}$, say $\{Sq^{2^i} \mid i \geq 0\}$. In $\mathrm{Ext}^1(\mathbb{F}_2,\mathbb{F}_2)$, the linear dual of $I\mathcal{A}/(I\mathcal{A})^2$, we let

h_i be dual to Sq^{2^i}. (The name stems from the fact that h_i detects the *Hopf map* of degree $2^i - 1$ when such a map exists, i.e., for $i \le 3$.)

The theorem then says, in particular, that if x is dual to g_j, then $h_i x$ is the sum of the $\mathcal{A}$-generators of C_{s+1} whose differential contains the term $Sq^{2^i} g_j$. Applying this to the minimal resolution in Section 1.5 gives all the products shown there. For example, the differentials $d(\eta_i) = Sq^{2^i} \iota_0$ essentially define the h_i. The differential $d(\eta_{00}) = Sq^1 \eta_0$ similarly justifies the name h_0^2 for the dual of this generator. More interestingly, in $C_{2,4}$ we find the generator η_{11} whose dual is h_1^2. The differential $d(\eta_{11})$ must be either $Sq^2\eta_1 + Sq^3\eta_0$ or $Sq^2\eta_1 + Sq^{0,1}\eta_0$. It cannot be simply $Sq^2\eta_1$ as this is not a cycle. Since Sq^3 is not indecomposable, the term $Sq^3\eta_0$ does not affect the product structure: we do not obtain another way of writing h_1^2 as a product, as we do from each of the three terms in

$$d(\eta_{111}) = Sq^4\eta_{00} + Sq^2\eta_{11} + Sq^1\eta_{02}$$

which tell us that $h_2(h_0^2) = h_1(h_1^2) = h_0(h_0h_2)$. However, the term $Sq^3\eta_0$ does affect the Massey products. If h_0 and h_1 lift to chain maps $\widetilde{h}_0$ and $\widetilde{h}_1$, respectively, then it is easy to check that the null-homotopy $H : \widetilde{h}_1 \circ \widetilde{h}_0 \ \simeq\ 0$ maps $C_2 \longrightarrow C_0$ by $\eta_{00} \mapsto 0$, $\eta_{11} \mapsto \eta_0$, since $\widetilde{h}_0(\eta_{11}) = Sq^1(\eta_1)$, so that $\widetilde{h}_1\widetilde{h}_0(\eta_{11}) = Sq^1(\iota_0) = d(\eta_0)$. In Section 2.6 we will see that the Massey product $\langle h_0, h_1, h_0\rangle$ contains the cocycle $h_0 \circ H$. Since $h_0 \circ H(\eta_{11}) = 1$, we see that

$$\langle h_0, h_1, h_0\rangle = h_1^2$$

perfectly reflecting the Toda bracket

$$\langle 2, \eta, 2\rangle = \eta^2$$

in $\pi_*(S)$.

2.5. Diagrammatic methods in module theory

Algebraic topologists have long represented cell complexes by graphs in which each vertex represents a cell and each edge is labelled by the attaching map of the cell (or its top dimensional component). Another variant is really a diagram of the (mod p) cohomology, with each vertex representing an $\mathbb{F}_p$ summand and the labelled edges corresponding to the action of a generating set for the Steenrod algebra or some subalgebra.

Representation theorists have formalized and greatly extended this idea with the notion of a *quiver*. In this more general version, vertices can now represent a variety of different modules and edges can be labelled by, for example, elements of the Ext module for the source and target.

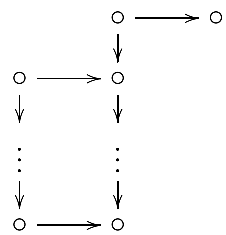
Some of the calculations relevant to connective K-theory can be given in quite clear form using diagrams of this sort. It seems likely that these methods will become more important in topology, so we have chosen to give a simple example here. The reader who wishes to learn more is encouraged to consult Benson [**7**, Ch. 4] or Carlson [**13**].

We will calculate the cohomology ring $H^*(Z/p, \mathbb{F}_p) = \mathrm{Ext}_{\mathbb{F}_p[Z/p]}(\mathbb{F}_p, \mathbb{F}_p)$. The Yoneda product turns out to be an easy and painless way to do this, in contrast to the effort required to produce a diagonal map and compute the ring structure from the it. The graphical representation of the resolutions and cocycles will be convenient, but not essential in this application. However, it serves as a simple example with which to introduce the method.

The group ring $A = \mathbb{F}_p[Z/p]$ can be identified with $\mathbb{F}_p[T]/(T^p)$, where $T = \tau - 1$, if $Z/p = <\tau \mid \tau^p = 1>$. We will represent A-modules by graphs in which each vertex is an $\mathbb{F}_p$ and edges are multiplications by T which map the corresponding $\mathbb{F}_p$'s isomorphically. Thus the extension

$$0 \longrightarrow IA \longrightarrow A \longrightarrow \mathbb{F}_p \longrightarrow 0$$

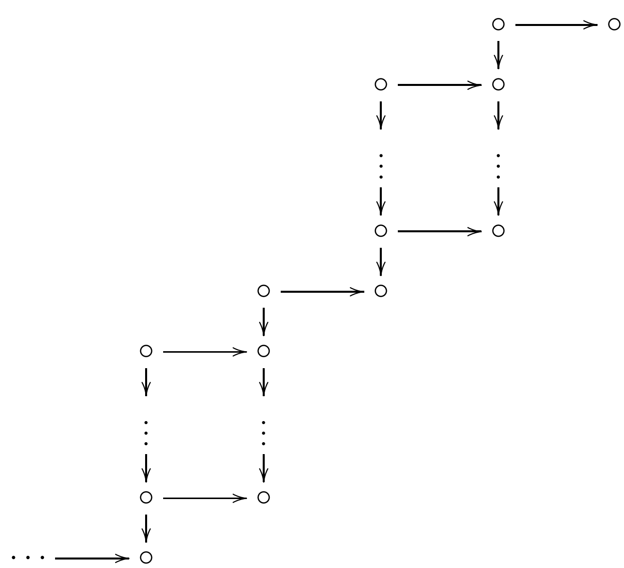
is graphically represented as



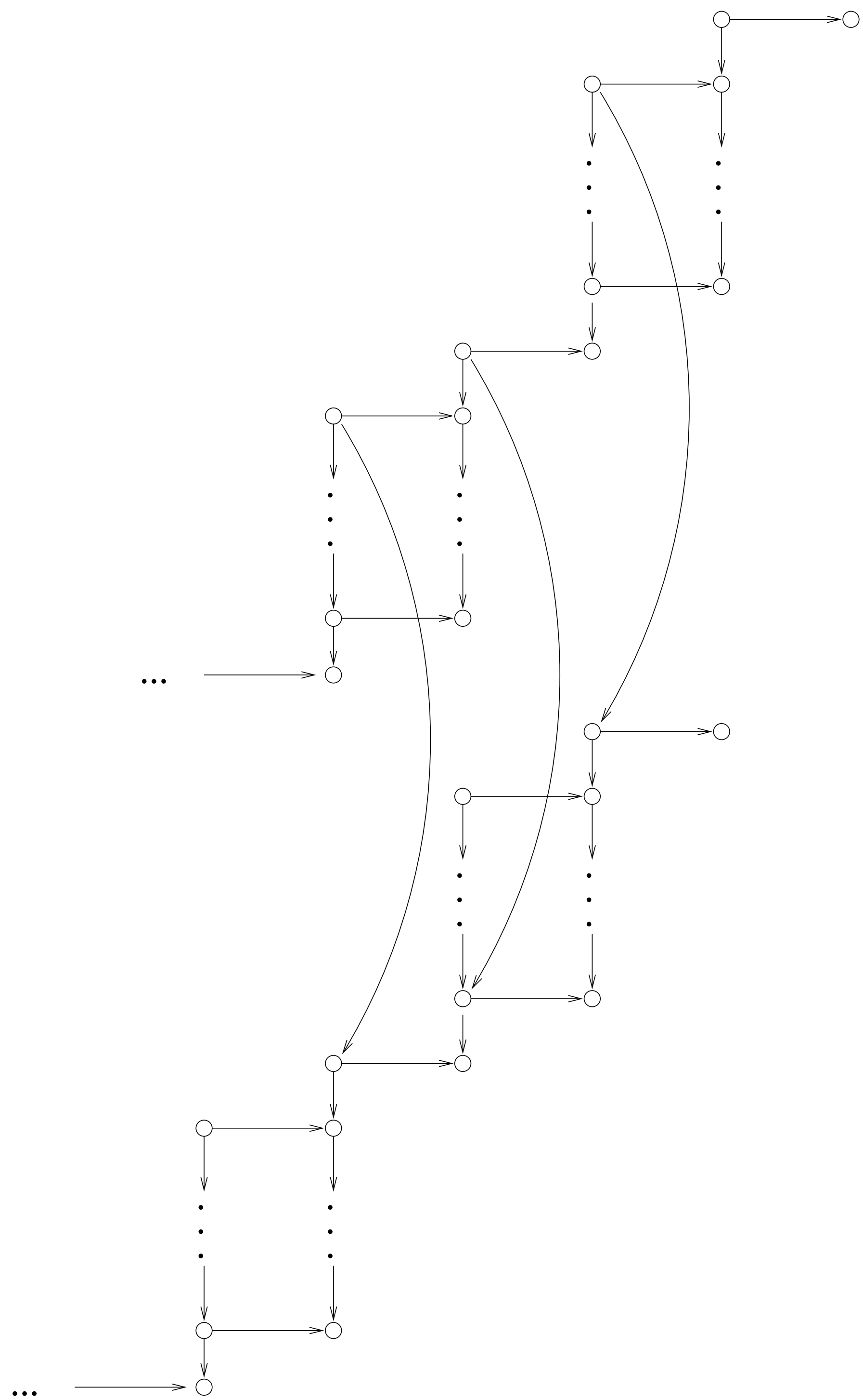
The horizontal arrows show the module homomorphisms and the vertical arrows represent multiplication by T in each of the three modules displayed. The middle module is p dimensional over $\mathbb{F}_p$. In it, the top vertex represents 1, the next T, etecetera, until the bottom one, which represents T^{p-1}. Since this is annihilated by T, there is no arrow from this vertex. The left module is $p-1$ dimensional, as can be deduced from the fact that its height is one less than that of the middle module.

We obtain a free resolution of $\mathbb{F}_p$ over A which is periodic of period 2, alternating multiplication by T and by T^{p-1}:

$$\cdots \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{F}_p$$



It follows that each $H^i(Z/p, \mathbb{F}_p) = \mathbb{F}_p$. To determine the ring structure, let x and y be nonzero elements of H^1 and H^2 respectively. Consider the chain map induced by x:



It follows immediately that, if $p = 2$ then multiplication by x is an isomorphism

$$H^i(Z/2, \mathbb{F}_2) \longrightarrow H^{i+1}(Z/2, \mathbb{F}_2)$$

for all i. Accordingly, $H^*(Z/2, \mathbb{F}_2) = \mathbb{F}_2[x]$, the polynomial algebra on x. If $p > 2$ then evidently $x^2 = 0$ and multiplication by x is an isomorphism

$$H^{2i}(Z/p, \mathbb{F}_p) \longrightarrow H^{2i+1}(Z/p, \mathbb{F}_p)$$

for all i. Similarly, the reader is encouraged to construct the chain map corresponding to multiplication by y, which will show that multiplication by y is an isomorphism

$$H^{i}(Z/p, \mathbb{F}_p) \longrightarrow H^{i+2}(Z/p, \mathbb{F}_p)$$

for all i. (This is practically evident in the periodicity of the resolution, of course.) Accordingly, $H^*(Z/p, \mathbb{F}_p) = E[x] \otimes \mathbb{F}_p[y]$, the tensor product of the exterior algebra on x and the polynomial algebra on y.

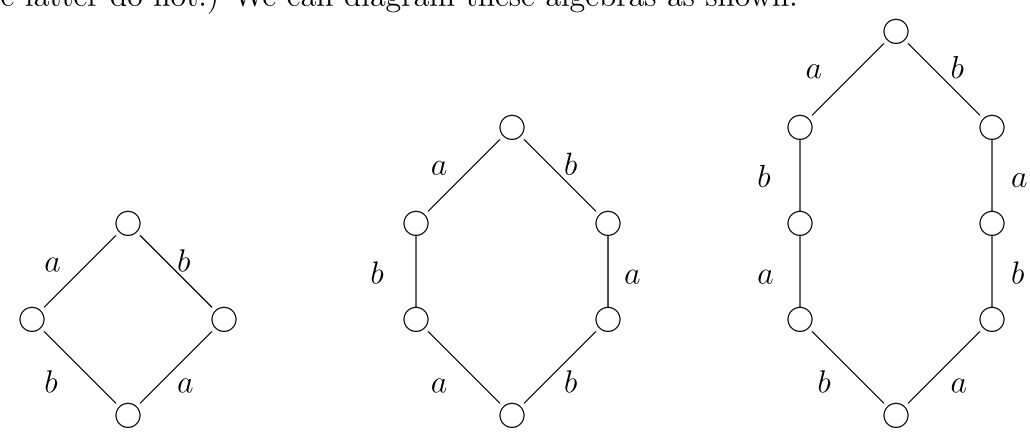
The dihedral algebras are another nice family of examples. Let k be a field of characteristic 2 and define k-algebras by

$$\begin{aligned} \mathcal{D}_{2i} &= k\langle a, b \mid a^2 = b^2 = 1, (ab)^i = (ba)^i\rangle\,, and \\ \mathcal{D}_{2i+1} &= k\langle a, b \mid a^2 = b^2 = 1, (ab)^i a = (ba)^i b\rangle\ . \end{aligned}$$

It is easy to see that $\mathcal{D}_i$ is $2i$-dimensional as a k-vector space, and that $\mathcal{D}_{2^n} = k[D_{2^{n+1}}]$, the group algebra of the dihedral group

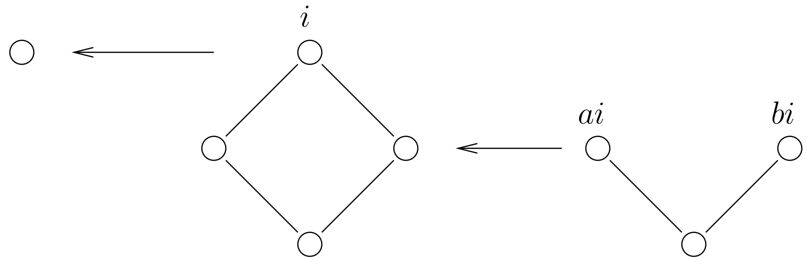
$$D_{2^{n+1}} = \langle x, y \mid x^2 = y^2 = (xy)^{2^n}\rangle$$

of order 2^{n+1}, by the isomorphism $a = x + 1$, $b = y + 1$. (Note that the other $\mathcal{D}_{2i}$ are not the group algebras of the groups D_{4i} since the former have nilpotent augmentation ideal and the latter do not.) We can diagram these algebras as shown:

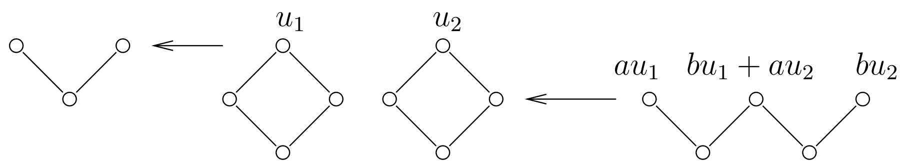


$\mathcal{D}_2$ $\mathcal{D}_3$ $\mathcal{D}_4$

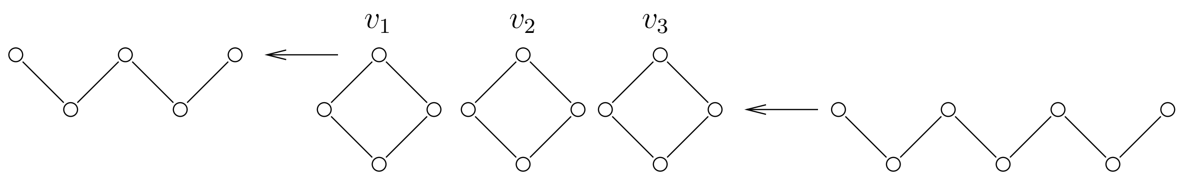
A free resolution for k over $\mathcal{D}_2$ is given by splicing the short exact sequences



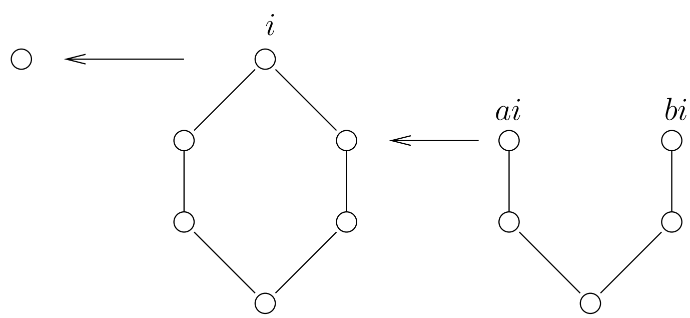
where $i \mapsto 1$,



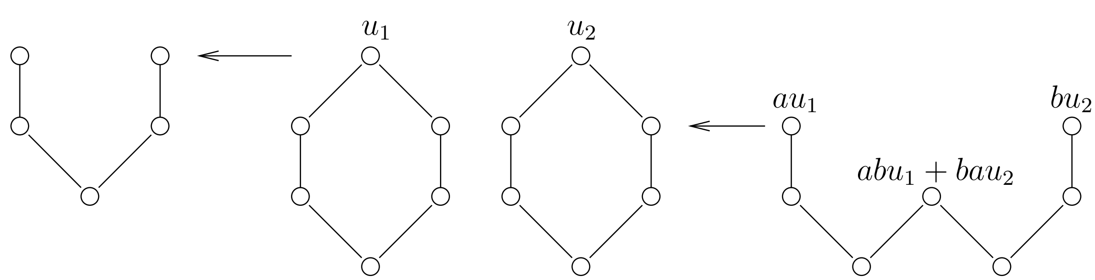
where $u_1 \mapsto ai$ and $u_2 \mapsto bi$,



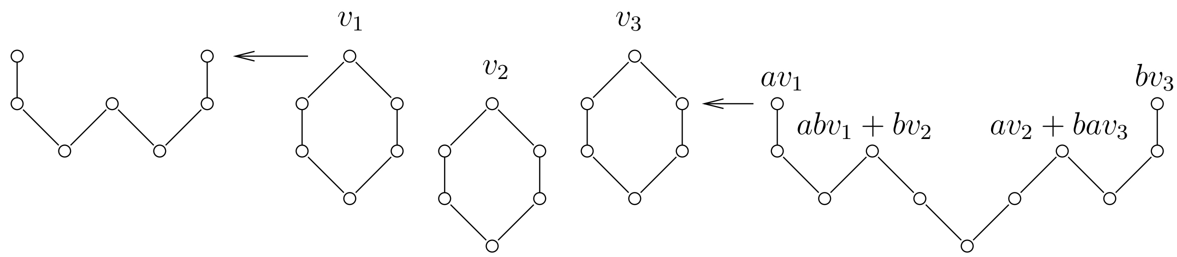
where $v_1 \mapsto au_1$, $v_2 \mapsto bu_1 + au_2$, and $v_3 \mapsto bu_2$, etc. It should be evident how to continue this. Similarly, we can resolve k over $\mathcal{D}_3$ by splicing the short exact sequences



where $i \mapsto 1$,



where $u_1 \mapsto ai$ and $u_2 \mapsto bi$,



where $v_1 \mapsto au_1$, $v_2 \mapsto abu_1 + bau_2$, and $v_3 \mapsto bu_2$, etc.

For both of these resolutions, we define cocycles $x : C_1 \longrightarrow k$ and $y : C_1 \longrightarrow k$ by

$$x: \begin{array}{ccc} u_1 & \mapsto & 1 \\ u_2 & \mapsto & 0 \end{array} \qquad\qquad y: \begin{array}{ccc} u_1 & \mapsto & 0 \\ u_2 & \mapsto & 1 \end{array}$$

and $w : C_2 \longrightarrow k$ by

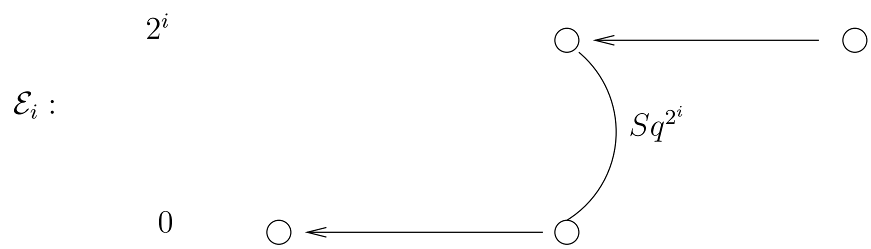
$$w: \begin{array}{ccc} v_1 & \mapsto & 0 \\ v_2 & \mapsto & 1 \\ v_3 & \mapsto & 0. \end{array}$$

For $\mathcal{D}_2$, it will transpire that $w = xy$, but for $\mathcal{D}_i$, $i > 2$, w is indecomposable.

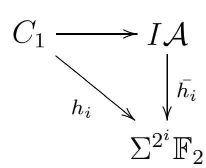
It should be clear from these how the general case behaves. The reader is invited to continue this in Exercise 2.5.5. (Note that we have transposed the a's and b's in the copy of $\mathcal{D}_3$ generated by v_2, to make the diagrams work more cleanly. Similar transpositions will be helpful for $\mathcal{D}_i$ when i is odd.)

Here are some exercises using these ideas.

EXERCISE 2.5.1. Show that an extension representing $h_i \in \mathrm{Ext}_{\mathcal{A}}^{1,2^i}$ is



by computing the pushout of the cocycle



where $\bar{h_i}(Sq^{2^j}) = \delta_{ij}$.

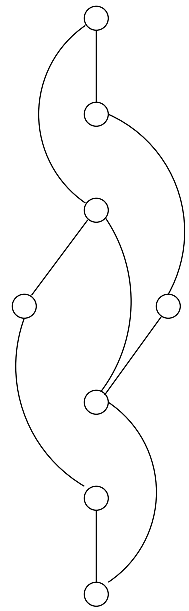
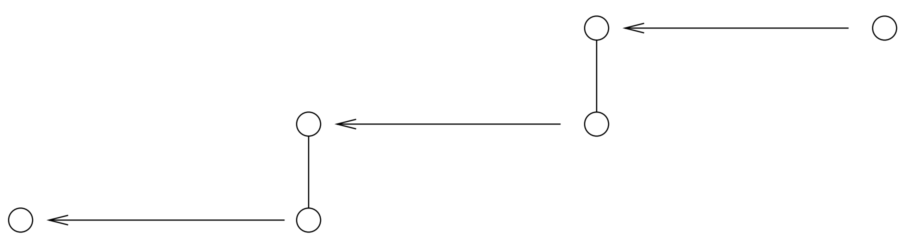


FIGURE 2.1. $\mathcal{A}(1) = < Sq^1, Sq^2 > \subset \mathcal{A}$. Straight lines represent Sq^1 and curved lines, Sq^2.

EXERCISE 2.5.2. Work over $\mathcal{A}(1)$ (see Figure 2.1) rather than $\mathcal{A}$ for simplicity. Note that the Yoneda composite of $\mathcal{E}_0$ with itself is

$\mathcal{E}_0 \circ \mathcal{E}_0$:



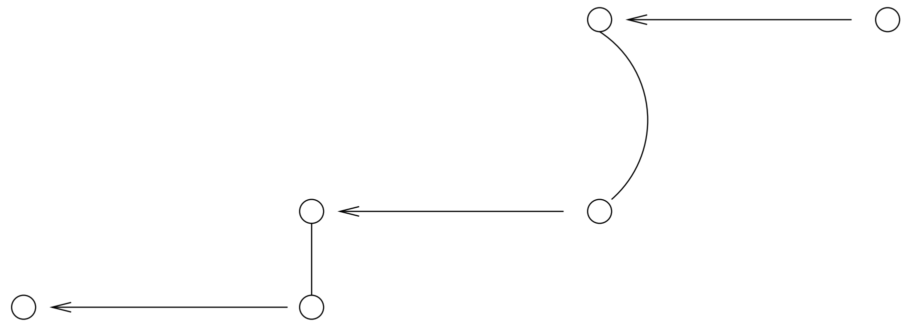
Compute the pushout

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow M_0 \longleftarrow M_1 \longleftarrow \Sigma^2\mathbb{F}_2 \longleftarrow 0$$

of the cocycle h_0^2, and show that $M_0 = \mathcal{A}(1)$. Compute the map (which is an equivalence of extensions) from the pushout to $\mathcal{E}_0 \circ \mathcal{E}_0$.

Exercise 2.5.3. Again, let us work over $\mathcal{A}(1)$ for simplicity. Show that the Yoneda composite

$\mathcal{E}_0 \circ \mathcal{E}_1$:



though not evidently trivial, is nonetheless a trivial extension, by showing it is equivalent to one of the form

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow \mathbb{F}_2 \oplus M \longleftarrow M \oplus \Sigma^3\mathbb{F}_2 \longleftarrow \Sigma^3\mathbb{F}_2 \longleftarrow 0,$$

which is evidently the Yoneda composite of two split extensions, and is therefore trivial.

Hint: The cocycle h_0h_1 is a coboundary. Use the factorization $C_2 \longrightarrow C_1 \longrightarrow \Sigma^3\mathbb{F}_2$ of h_0h_1 to show that the pushout of h_0h_1 has the required form. Use the fact that it is a pushout to map it to $\mathcal{E}_0 \circ \mathcal{E}_1$.

Find a still simpler trivialization

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow M_0 \longleftarrow M \oplus \Sigma^3\mathbb{F}_2 \longleftarrow \Sigma^3\mathbb{F}_2 \longleftarrow 0$$

which is the Yoneda composite of an extension $0 \longleftarrow \mathbb{F}_2 \longleftarrow M_0 \longleftarrow M \longleftarrow 0$ and a split extension $0 \longleftarrow M \longleftarrow M \oplus \Sigma^3\mathbb{F}_2 \longleftarrow \Sigma^3\mathbb{F}_2 \longleftarrow 0$, where M is 2 dimensional over $\mathbb{F}_2$.

Remark 2.5.4. In the preceding two exercises you may wish to use the minimal resolution of $\mathcal{A}(1)$ computed in Chapter 9 (Figure **??**).

Exercise 2.5.5. . Using the beginnings of the minimal resolutions for $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ above as models, compute minimal resolutions of k over $\mathcal{D}_i$ for all i. Compute chain maps induced by the cocycles x, y, and w, defined above (and their generalizations to arbitrary i), and use them to show that

$$H^*(D;k) = \begin{cases} k[x,y] & n = 2 \\ k[x,y,w]/(xy) & n > 2. \end{cases}$$

2.6. Massey products and Toda brackets

Massey products and Toda brackets are examples of 'secondary compositions': elements in homology or homotopy which are formed using null-homotopies. When dealing with algebras which have many zero divisors, these secondary compositions play an important role. Both the homotopy groups of spheres and the cohomology of the Steenrod algebra fall into this group.

Massey products can be defined in the homology of a differential algebra. In a suitably general form ('matrix Massey products') and a wide variety of situations, they allow the construction of all elements in homology as Massey products of elements in homological degree 1. For a definitive treatment, see May **??**. Here we shall use only the simplest case, which is sufficient to exemplify the idea, and to produce the elements we shall study.

Toda brackets can be defined in triangulated categories such as the stable homotopy category or the category of chain complexes. The latter case includes $\mathrm{Ext}_A(k,k)$, the cohomology of a Hopf algebra A. Here we can either define Toda brackets of chain maps, or define Massey products since $\mathrm{Ext}_A(k,k) = H(\mathcal{C})$, where $\mathcal{C}$ is an appropriate differential algebra. Naturally, these two constructions will agree in this case. As we will discuss in Chapter 10, efficiency considerations make the Toda bracket of chain maps the method of choice for actual calculations, but both descriptions are useful for theoretical considerations.

Let us start with Massey products. Suppose $\mathcal{C}$ is a differential algebra, and let $x_i \in H(\mathcal{C})$ for $i = 1,2,3$ satisfy $x_1x_2 = 0$ and $x_2x_3 = 0$. Choose cycles $a_i \in \mathcal{C}$ representing the x_i. Then there are elements $a_{12}, a_{23} \in \mathcal{C}$ such that $d(a_{12}) = a_1a_2$ and $d(a_{23}) = a_2a_3$. Define $\bar{a} = (-1)^{|a|}a$, for $a \in \mathcal{C}$. Then

$$\begin{aligned} d(a_{12}a_3 - \bar{a}_1a_{23}) &= d(a_{12})a_3 - (-1)^{|a_1|}\bar{a}_1 d(a_{23}) \\ &= a_1a_2a_3 - a_1a_2a_3 \\ &= 0 \end{aligned}$$

so $a_{12}a_3 - \bar{a}_1a_{23}$ defines an element in $H(\mathcal{C})$. We define the Massey product

$$\langle x_1, x_2, x_3\rangle = \{a_{12}a_3 - \bar{a}_1a_{23} \mid d(a_{ij}) = a_ia_j\}.$$

Since the choice of a_{12} and a_{23} can be altered by any cycle, the *indeterminacy* of $\langle x_1, x_2, x_3\rangle$, defined by

$$In(\langle x_1, x_2, x_3\rangle) = \{a - b \mid a, b \in \langle x_1, x_2, x_3\rangle\}$$

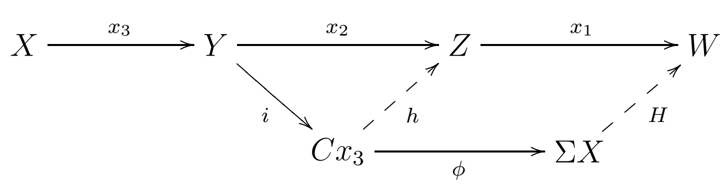
can be described as

$$In(\langle x_1, x_2, x_3\rangle) = x_1H^{|x_2x_3|}(\mathcal{C}) + H^{|x_1x_2|}(\mathcal{C})x_3.$$

To define a Toda bracket, let us start with maps

$$X \xrightarrow{x_3} Y \xrightarrow{x_2} Z \xrightarrow{x_1} W$$

such that $x_1x_2 \simeq 0$ and $x_2x_3 \simeq 0$. Then we have an extension $h : Cx_3 \longrightarrow Z$ of x_2 over the cofiber of x_3,



and, since $x_1hi \simeq x_1x_2 \simeq 0$, there is an extension $H : \Sigma X \longrightarrow W$ of x_1h over ϕ. The *Toda bracket* $\langle x_1, x_2, x_3\rangle$ is the set of all such H.

In the category of topological spaces, where ΣX is the union of two cones on X, we can produce maps H by putting

$$CX \xrightarrow{x_{23}} Z \xrightarrow{x_1} W$$

on one cone, and

$$CX \xrightarrow{Cx_3} CY \xrightarrow{x_{12}} W$$

on the other, where $x_{12} : x_1x_2 \simeq 0$ and $x_{23} : x_2x_3 \simeq 0$ are null-homotopies.

In the category of chain complexes, the cofiber of x_3 is $(Cx_3)_n = X_n \oplus Y_{n+1}$ with differential

$$d = \begin{pmatrix} d & 0 \\ (-1)^n (x_3)_n & d \end{pmatrix}.$$

A chain map $h : Cx_3 \longrightarrow Z$ extending x_2 must then be x_2 on the Y summand and a null-homotopy of $x_2 x_3$ on the X summand. Similarly, the cofiber of $i : Y \longrightarrow Cx_3$ has an additional summand Y_{n-1} in degree n, and the extension H restricted to this summand is a null-homotopy of $x_1 x_2$.

This makes the analogy with the Massey product evident. We can make it more precise as follows. Replace the differential algebra $\mathcal{C}$ by the differential algebroid (i.e., algebra with *many objects*) of all graded homomorphisms between our chain complexes. Let us write $\mathrm{Hom}(X, Y)$ for graded homomorphisms from X to Y, *not* required to commute with the differential. We can define a derivation $\delta : \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X, Y)$ by $\delta(x) = dx - \bar{x}d$. Write composition as juxtaposition, as usual. The following lemmas are simple calculations.

LEMMA 2.6.1. *The homomorphism δ satisfies:*

(1) $\delta\delta x = 0$
(2) $\delta(ab) = \delta(a)b + \bar{a}\delta(b)$
(3) $\overline{\delta(x)} = xd - d\bar{x}$

LEMMA 2.6.2. $\delta(x) = 0$ *iff x is a chain map.*

LEMMA 2.6.3. *h is a null homotopy, $h : x \simeq 0$, iff $\delta(h) = x$.*

We therefore have

PROPOSITION 2.6.4. *The homology of* $\mathrm{Hom}(X, Y)$ *with respect to δ,* $H(\mathrm{Hom}(X, Y), \delta)$ *is the set of chain homotopy equivalence classes of chain maps,* $[X, Y]$.

Now the Massey product definition $\langle x_1, x_2, x_3 \rangle = \{a_{12}a_3 - \bar{a}_1 a_{23} \mid [a_i] = x_i, \delta(a_{ij}) = a_i a_j\}$ has an explicit interpretation in terms of chain maps and chain null-homotopies which is easily verified to be equivalent to the Toda bracket definition as outlined above.

A surprise emerges from this. Since a chain null-homotopy has no nonzero component mapping into homological degree 0, the cocycle corresponding to $a_{12}a_3 - \bar{a}_1 a_{23}$ does not depend on a_{12}. Thus, we need only calculate the null-homotopy a_{23} and compose with a_1 to compute the Toda bracket (Massey product) $\langle x_1, x_2, x_3 \rangle$. This is counterintuitive at first encounter, in that it appears to say that the null-homotopy of $a_1 a_2$ makes no difference, and in fact might not even be needed. However, this is not so. The composite $\bar{a}_1 a_{23}$ is not a chain map, even though its component mapping into degree 0 suffices to determine a cocycle. When we lift this cocycle to a chain map, the additional terms needed to convert $\bar{a}_1 a_{23}$ into a chain map produce the missing piece, $a_{12}a_3$. Calculationally, this is a great convenience: from a single null-homotopy, a_{23}, we are able to determine all Massey products $\langle x_1, x_2, x_3 \rangle$.

An additional simplification occurs when x_1 has homological degree 1. The same sort of argument as in Section 2.4 shows that we may calculate $\langle x_1, x_2, x_3 \rangle$ directly from the chain map x_3, without need of the null-homotopy, just as we were able to compute products $x_1 x_2$ where x_1 has homological degree 1, without having to compute the chain map induced by x_2.

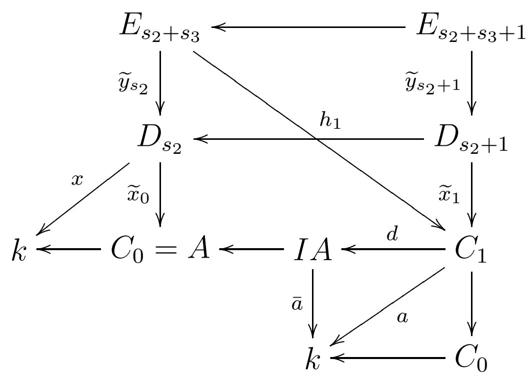
Adopt the notation of Section 2.4: let $a \in \mathrm{Ext}^1_A(k,k)$, and let $x \in \mathrm{Ext}^{s_2}(M,k)$ and $y \in \mathrm{Ext}^{s_3}(N,M)$, so that $\langle a,x,y\rangle \in \mathrm{Ext}^{s_2+s_3}(N,k)$. Let $C \longrightarrow k$, $D \longrightarrow M$ and $E \longrightarrow N$ be resolutions of M and N. Let $h : xy \simeq 0$, with degree 1 component $h_1 : E_{s_2+s_3} \longrightarrow C_1$.

THEOREM 2.6.5. *If* $g \in E_{s_2+s_3}$ *write* $\tilde{y}_{s_2}(g) = \sum_i a_i g_i \in D_{s_2}$. *Then*

$$\langle a,x,y\rangle(g) = \sum_i \bar{a}(a_i)x(g_i)$$

Proof: We compute:

$$\begin{aligned}\langle a,x,y\rangle(g) &= ah_1(g)\\ &= \bar{a}dh_1(g)\\ &= \bar{a}\widetilde{x}_0\widetilde{y}_{s_2}(g)\\ &= \sum_i \bar{a}(a_i)x(g_i)\end{aligned}$$



□

Here is what the theorem says in the cohomology of the Steenrod algebra. Recall that h_i is dual to Sq^{2^i}. If x is dual to an $\mathcal{A}$-generator g_j of the resolution, then the Massey product $\langle h_i, x, y\rangle$ is the sum of those $\mathcal{A}$-generators g such that $\widetilde{y}(g)$ contains the term $Sq^{2^i}g_j$. Linearity of the Massey product then determines all Massey products $\langle h_i,x,y\rangle$ from the chain map $\widetilde{y}$.

In Figure 2.2 we give the chain maps induced by h_0 and h_1 in low degrees, and in Figure 2.3 we give the Massey products which follow from these. We follow the customary practice of writing Massey products as elements rather than singleton sets when their indeterminacy is 0.

For example, since $\widetilde{h_0}(\eta_{11}) = Sq^1\eta_1$, and h_1 is dual to η_1, we get that $\langle h_0,h_1,h_0\rangle$ is dual to η_{11}s, and hence equal to h_1^2.

Note that the presence of a term $Sq^{2^i}g_j$ in $\widetilde{y}(g)$ does not imply that the Massey product $\langle h_i,x,y\rangle$ exists. The products h_ix and xy may not be zero. For example, $\widetilde{h_0}(\eta_{13})$ contains the term $Sq^8\eta_0$, but the Massey product $\langle h_3,h_0,h_0\rangle$ fails to exist on two counts: both h_0h_3 and h_0^2 are nonzero.

x	$\widetilde{h_0}(x)$	$\widetilde{h_1}(x)$
η_0	ι	0
η_1	0	ι
η_2	0	0
η_3	0	0
η_{00}	η_0	0
η_{11}	$(1)\eta_1$	η_1
η_{02}	η_2	$(2)\eta_0 + (1)\eta_1$
η_{22}	$(3)\eta_2$	$(5)\eta_0 + (2)\eta_2$
η_{03}	$(7)\eta_0 + (6)\eta_1 + \eta_3$	$((6) + (0,2))\eta_0 + (5)\eta_1 + (3)\eta_2$
η_{13}	$((8) + (2,2))\eta_0 + ((4,1) + (0,0,1))\eta_1$	$(7)\eta_0 + \eta_3$
η_{000}	η_{00}	0
η_{111}	$(1)\eta_{11} + \eta_{02}$	η_{11}
η_{003}	$(5)\eta_{11} + \eta_{03}$	$((4) + (1,1))\eta_{11}$
γ_0	$(8)\eta_{00} + (5)\eta_{02} + (2)\eta_{22}$	$(7)\eta_{00} + (4)\eta_{02} + (1)\eta_{22}$
η_{222}	$((9) + (3,2))\eta_{00} + (0,0,1)\eta_{11} + (3)\eta_{22} + (1)\eta_{13}$	$(8)\eta_{00} + (5)\eta_{02} + (2)\eta_{22} + \eta_{13}$
η_{0^4}	η_{000}	0
η_{0003}	$(7)\eta_{000} + ((4) + (1,1))\eta_{111} + \eta_{003}$	$(6)\eta_{000} + ((3) + (0,1))\eta_{111}$
γ_{00}	$(3,2)\eta_{000} + ((6) + (0,2))\eta_{111} + (1)\gamma_0$	$(8)\eta_{000} + (5)\eta_{111} + \gamma_0$
η_{0^5}	η_{0^4}	0
ρ_1	$(9)\eta_{0^4} + (2)\eta_{0003}$	$(8)\eta_{0^4} + (1)\eta_{0003}$
ρ_2	$(11)\eta_{0^4} + (4)\eta_{0003} + (2)\gamma_{00}$	$(1)\gamma_{00}$
η_{0^6}	η_{0^5}	0
ρ_{11}	$(10)\eta_{0^5} + (1)\rho_1$	$(9)\eta_{0^5} + \rho_1$
ρ_{02}	$(11)\eta_{0^5} + \rho_2$	$(10)\eta_{0^5} + (1)\rho_1$
η_{0^7}	η_{0^6}	0
ρ_{111}	$(11)\eta_{0^6} + (1)\rho_{11} + \rho_{02}$	$(10)\eta_{0^6} + \rho_{11}$

FIGURE 2.2. Chain maps lifting h_0 and h_1 in the resolution of Figure 1.2

The Massey products fall into three groups. Those in the first group all follow from a general formula:

$$y(x \cup_1 x) = ySq^{s-1}(x) \in \langle x, y, x \rangle$$

if $x \in \mathrm{Ext}^s$. We will discuss the cup-i construction and the Steenrod operations which apply to the cohomology of a cocommutative Hopf algebra such as the Steenrod algebra in the next two chapters, and prove this formula (Theorem **??**). Since $h_i \in \mathrm{Ext}^1$ and $Sq^0(h_i) = h_{i+1}$, we get

$$h_{i+1}y \in \langle h_i, y, h_i \rangle.$$

All the Massey products in the first group are of this form.

The Massey products in the second group have the form

$$\langle h_3, h_0^4, x \rangle$$

or can be reduced to it by various identities for Massey products, such as

$$\langle h_3, h_0^4, x \rangle = \langle x, h_0^4, h_3 \rangle = \langle x, h_0^3, h_0 h_3 \rangle = \langle x, h_0^3 h_3, h_0 \rangle.$$

$\langle h_0, h_1, h_0\rangle$	$=$	h_1^2
$\langle h_0, h_1^2, h_0\rangle$	$=$	$\{0, h_1^3\}$
$\langle h_0, h_1h_3, h_0\rangle$	$=$	$h_1^2h_3 = h_2^3$
$\langle h_0, c_0, h_0\rangle$	$=$	h_1c_0
$\langle h_0, P^1h_1, h_0\rangle$	$=$	$h_1P^1h_1$
$\langle h_0, h_1P^1h_1, h_0\rangle$	$=$	$\{0, h_1^2P^1h_1\}$
$\langle h_1, h_0, h_1\rangle$	$=$	h_0h_2
$\langle h_1, h_2, h_1\rangle$	$=$	h_2^2
$\langle h_1, h_2^3, h_1\rangle$	$=$	$\{0, h_2^3\}$
$\langle h_1, h_0^3h_3, h_0\rangle$	$=$	P^1h_1
$\langle h_2, h_0^3h_3, h_0\rangle$	$=$	P^1h_2
$\langle h_0, h_0^3h_3, h_1\rangle$	$=$	P^1h_1
$\langle h_3, h_0^4, h_1\rangle$	$=$	P^1h_1
$\langle h_1, h_2^2, h_0\rangle$	$=$	c_0
$\langle h_2, h_1^3, h_0\rangle$	$=$	$\{0, h_0^3h_3\}$
$\langle h_2, h_0h_2, h_1\rangle$	$=$	c_0
$\langle h_0, h_2^2, h_1\rangle$	$=$	c_0
$\langle h_1, h_1c_0, h_0\rangle$	$=$	P^1h_2

FIGURE 2.3. Massey products derived from the chain maps of Figure 2.2

This is one definition of Adams' periodicity operator:

$$P^1x = \langle h_3, h_0^4, x\rangle$$

if $h_0^4x = 0$.

The third group contains the rest. Note in particular that c_0, though indecomposable as a product, does have the Massey product descriptions

$$c_0 = \langle h_1, h_2^2, h_0\rangle = \langle h_1, h_2, h_0h_2\rangle = \langle h_1, h_0h_2, h_2\rangle$$

and similarly,

$$P^1h_2 = \langle h_0, h_1c_0, h_1\rangle.$$

We have included all the deductions which can be made from the chain maps in Figure 2.2. This is redundant, of course: since $c_0 = \langle h_1, h_2^2, h_0\rangle = \langle h_0, h_2^2, h_1\rangle$, it can be computed from either of $\widetilde{h_0}$ or $\widetilde{h_1}$, but we wanted the comparison between the chain maps and the Masssey products which can be deduced from them to be as straightforward as possible.

Finally, since there are no differentials in the Adams spectral sequence in these low dimensions, these Massey products give Toda brackets in the stable homotopy groups of spheres, by Moss' convergence theorem ([**?**]). For example,

$$\begin{aligned}\langle 2, \eta, 2\rangle &= \eta^2 \text{ and}\\ \langle \eta, 2, \eta\rangle &= 2\nu.\end{aligned}$$

CHAPTER 3

The Origins of Steenrod Operations

3.1. Where do Steenrod operations come from?

Of course, nowdays we simply say that $\mathcal{A} = H^*H = [H, H]$, the endomorphism ring of mod p cohomology. But, how do we know what this algebra is? In fact, it was known before the representing objects, the Eilenberg-MacLane spectra, were constructed. Also, very similar algebras act on

- H_*QX, where $QX = \varinjlim(X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \cdots)$,
- $H^*\mathcal{E} = \mathrm{Ext}_{\mathcal{E}}(k, k)$, where $\mathcal{E}$ is a cocommutative Hopf algebra over a field k of positive characteristic, and
- the cohomology of restricted Lie algebras in positive characteristic,

among others.

<u>Answer</u>: A highly symmetric product plus the need to make choices which destroy that symmetry lead to operations which come from comparison of the choices. We will illustrate this with three examples.

3.2. The cup-i construction in the mod 2 cohomology of spaces

The diagonal map $\Delta : X \longrightarrow X \times X$, which is strictly cocommutative, is the geometric source of the cup product in cohomology. Cocommutativity means that if $\tau(x_1, x_2) = (x_2, x_1)$ is the transposition, then $\tau\Delta = \Delta$:

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ & \searrow^{\Delta} & \downarrow{\scriptstyle\tau} \\ & & X \times X \end{array}$$

If $C_*(\bullet)$ and $C^*(\bullet)$ represent singular chains and cochains, then the commutative triangle

$$\begin{array}{ccc} C^*(X) & \xleftarrow{\Delta_*} & C^*(X \times X) \\ & \nwarrow^{\Delta_*} & \uparrow{\scriptstyle\tau_*} \\ & & C^*(X \times X) \end{array} \qquad \text{induces} \qquad \begin{array}{ccc} H^*(X) & \xleftarrow{\mu} & H^*(X \times X) \\ & \nwarrow^{\mu} & \uparrow{\scriptstyle\tau^*} \\ & & H^*(X \times X) \end{array}$$

This is not quite a product in H^*X yet: we also need the Künneth isomorphism $H^*(X\times X) \cong H^*X \otimes H^*X$, and this is where the perfect (co)commutativity of Δ and Δ^* is destroyed. On the chain level, we have

$$C_*X \xrightarrow{\Delta_*} C_*(X \times X) \underset{\simeq}{\xrightarrow{AW}} C_*X \otimes C_*X$$

and thereby obtain an operation on cochains,

$$(a \cup b)(x) = (a \otimes b)(AW(\Delta_*(x))).$$

Here AW is a natural chain equivalence, for example, the *Alexander-Whitney* map. Such maps are unique up to homotopy (see [**23**, pp. 238-248] for a nice discussion), but are only homotopy commutative, not exactly commutative. For example, in order to write the diagonal 1-simplex in the unit square as a sum of product simplices, we must either choose the left edge plus the top edge, or the bottom edge plus the right edge, or some other linear combination of these whose coeffficents sum to 1. For commutativity, we would need both coefficients to be the same, and this cannot be done unless 2 is invertible. However, the resulting operation is homotopy commutative, $\cup\tau \simeq \cup$; in other words, the triangle

$$\begin{array}{ccc} C^*X \otimes C^*X & \xrightarrow{\cup} & C^*X \\ \downarrow{\scriptstyle \tau} & \nearrow_{\cup} & \\ C^*X \otimes C^*X & & \end{array}$$

homotopy commutes. Thus, we have a chain homotopy $\cup_1 : \cup\tau \simeq \cup$:

$$\begin{aligned} d \cup_1 (a \otimes b) + \cup_1 d(a \otimes b) &= \cup\tau(a \otimes b) - \cup(a \otimes b) \\ &= b \cup a - a \cup b \end{aligned}$$

so that

$$\begin{aligned} d \cup_1 (a \otimes a) + \cup_1 d(a \otimes a) &= a \otimes a - a \otimes a \\ &= 0 \end{aligned}$$

and if a is a cocycle, then $d(a \otimes a) = 0$ and so $d \cup_1 (a \otimes a) = 0$. This gives

$$Sq^{n-1}(a) := a \cup_1 a \in H^{2n-1}X$$

for $a \in H^n X$, the mod 2 cohomology of X.

Repeating this process leads to chain homotopies $\cup_{i+1} : \cup_i\tau \simeq \cup_i$ for each $i \geq 0$, and from these we obtain the Steenrod operations

$$Sq^{n-i}(a) := a \cup_i a \in H^{2n-i}X$$

in mod 2 cohomology.

3.3. The Dyer-Lashof operations in the homology of infinite loop spaces

Since $QX = \Omega Q\Sigma X = \Omega X_1$, we have a product, loop sum, in QX. In fact, we have two products, $(f, g) \mapsto f * g$ and $(f, g) \mapsto g * f$, where

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

But $QX = \Omega^2 Q\Sigma^2 X = \Omega^2 X_2$ as well. The product

$$(f * g)(t_1, t_2) = \begin{cases} f(2t_1, t_2) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2) & 1/2 \leq t_1 \leq 1, \end{cases}$$

and these two products, $f * g$ and $g * f$, are homotopic, as shown in Figure 3.1. There, the parts of the rectangle labelled f and g are mapped by f and g respectively after rescaling the box they are in to the unit square, and everything outside is mapped to the basepoint.

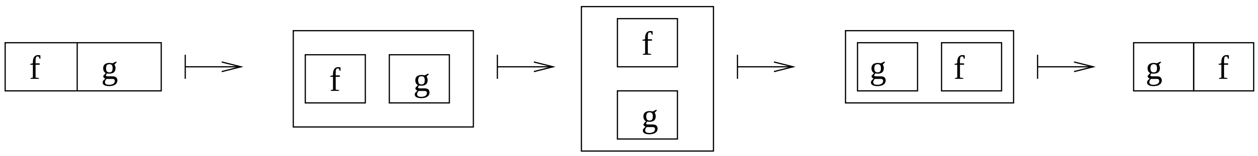


FIGURE 3.1. Homotopy between $f * g$ and $g * f$ in $\Omega^2 X_2$

In fact, there are two homotopies: in one the box containing f travels above that containing g, and in the other, it travels below that containing g (see Figure 3.2).

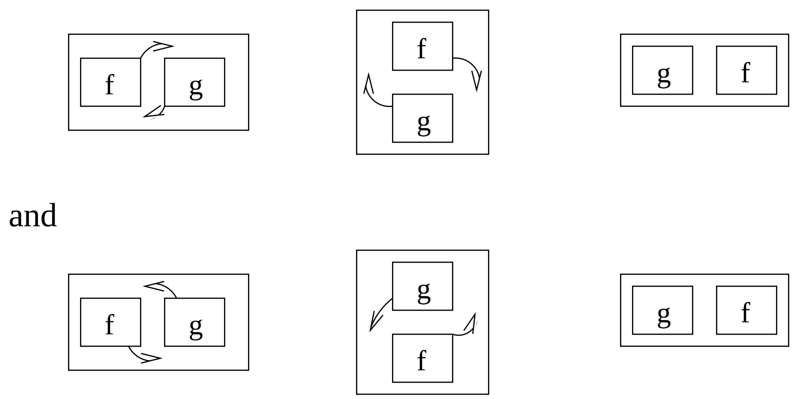


FIGURE 3.2. The two homotopies between $f * g$ and $g * f$

But $QX = \Omega^3 Q\Sigma^3 X = \Omega_3 X_3$, so these two homotopies are themselves homotopic: in a cube, we may rotate the path in which f travels above g into the path in which f travels below g, say with f's path travelling in front and that of g behind. Again, we have two choices: we could have let f's path travel behind and g's in front. In $QX = \Omega^4 Q\Sigma^4 X$, *these* two homotopies of homotopies are homotopic, etcetera, ad infinitum.

This infinite sequence of higher homotopies leads to Dyer-Lashof operations

$$Q^i : H_n QX \longrightarrow H_{n+i} QX$$

in the *homology* of QX with $Q^n(x) = x^2$, the Pontrjagin square. In fact, the Dyer-Lashof operations act on the homology of any infinite loop space E by exactly the same reasoning. Recall that an em infinite loop space E is a space for which there exist spaces E_i for each i such that $E \cong \Omega E_1 \cong \Omega^2 E_2 \cong \Omega^3 E_3 \cong \cdots$. For example, $E = QX$. The infinite loop spaces QX are special in the sense that the homology of QX is the free module over the Dyer-Lashof algebra generated by $H_* X$ in the appropriate sense: if $x \in H_n QX$ then $Q^i X = 0$ if $i < n$, and $Q^n x = x^2$, etc.

3.4. Steenrod operations in the cohomology of a cocommutative Hopf Algebra

Let A be a cocommutative Hopf algebra over $\mathbb{F}_p$. For example, A could be a group ring $\mathbb{F}_p[G]$ or the mod p Steenrod algebra $\mathcal{A}$.

If M and N are (left) A-modules, then $M \otimes N$ is naturally a (left) $A \otimes A$-module, and pullback along the diagonal map $\psi : A \longrightarrow A \otimes A$ converts it into an A-module. In formulas, $a(m \otimes n) = \sum a'm \otimes a''n$ if $\psi(a) = \sum a' \otimes a''$. Since A is cocommutative, $M \otimes N$ is naturally isomorphic to $N \otimes M$ by the obvious transposition.

Let

$$\mathcal{C} : \qquad 0 \longleftarrow \mathbb{F}_p \longleftarrow C_0 \longleftarrow C_1 \longleftarrow C_2 \longleftarrow \cdots$$

be a free resolution of $\mathbb{F}_p$ over A. Then $\mathcal{C} \otimes \mathcal{C}$ is also a free resolution of $\mathbb{F}_p \cong \mathbb{F}_p \otimes \mathbb{F}_p$. By the Comparison Theorem in homological algebra, there is a unique chain homotopy class of maps $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ covering the isomorphism $\mathbb{F}_p \cong \mathbb{F}_p \otimes \mathbb{F}_p$. Since A is cocommutative, $\tau\Delta$ and Δ are chain maps covering the same homomorphism, and are thus chain homotopic. (Note that if A were not cocommutative, then $\tau\Delta$ and Δ would be mapping to different A-modules, one with action $a(m \otimes n) = \sum a'm \otimes a''n$ and the other with action $a(m \otimes n) = \sum a''m \otimes a'n$.)

For example, suppose that $A = \mathbb{F}_2[Z/2]$ and that $\mathcal{C} = \mathcal{W}$ is the cellular chain complex for the standard $Z/2$-equivariant cell decomposition of S^∞. Explicitly, $\mathcal{W}$ is the graded differential A-module free on elements $e_i \in \mathcal{W}_i$, with differential $de_i = (1 + T)e_{i-1}$, where $Z/2 = \{1, T\}$, and with augmentation $\epsilon : \mathcal{W} \longrightarrow \mathbb{F}_2$, $\epsilon(e_0) = 1$ and $\epsilon(e_i) = 0$ otherwise.

If we start trying to construct the diagonal $\mathcal{W} \longrightarrow \mathcal{W} \otimes \mathcal{W}$, we can set $\Delta(e_0) = e_0 \otimes e_0$, but then $\Delta(e_1)$ must be $e_1 \otimes e_0 + Te_0 \otimes e_1$ or $e_0 \otimes e_1 + e_1 \otimes Te_0$, for example. No symmetrical choice for $\Delta(e_1)$ exists.

In general, we can repeat exactly the same sort of iterative construction we have made in the previous two examples. We have a chain homotopy $\Delta_1 : \tau\Delta \simeq \Delta$, etcetera. The algebra $\bar{\mathcal{A}}$ of operations which we get from this acts naturally on the cohomology $H^*A = \mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p)$ of any cocommutative Hopf algebra A, or more generally on $\mathrm{Ext}_A(M, N)$ if M is a coalgebra and N an algebra in the category of A-modules. It is an extension of the usual Steenrod algebra which acts on the cohomology of topological spaces by the monoid ring on an operation Sq^0:

$$0 \longrightarrow \mathbb{F}_2[Sq^0] \longrightarrow \bar{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow 0$$

(We will focus on the case $p = 2$ for simplicity here. The general case will be dealt with in the next chapter.) The element Sq^0 is central and maps to the identity operation in $\mathcal{A}$.

Since $\mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p)$ is bigraded, there are two indices affected by an operation, and there are two natural indexing conventions one could adopt. One is appropriate to cohomology operations and will agree with the usual indexing of Steenrod operations on topological spaces under the isomorphism between the cohomology of a group G and the cohomology of its classifying space BG, while the other is appropriate to homology operations and will agree with the Dyer-Lashof operations in the homology of an S-algebra (E_∞-ring spectrum in the older terminology) under the edge homomorphism of the Adams spectral sequence [**12**, p. 130]. We shall adopt the former here, since it turns out to be simpler to remember. In Figure 3.3 we show their location in the $(n, s) = (t - s, s)$ coordinates generally used for the Adams spectral sequence.

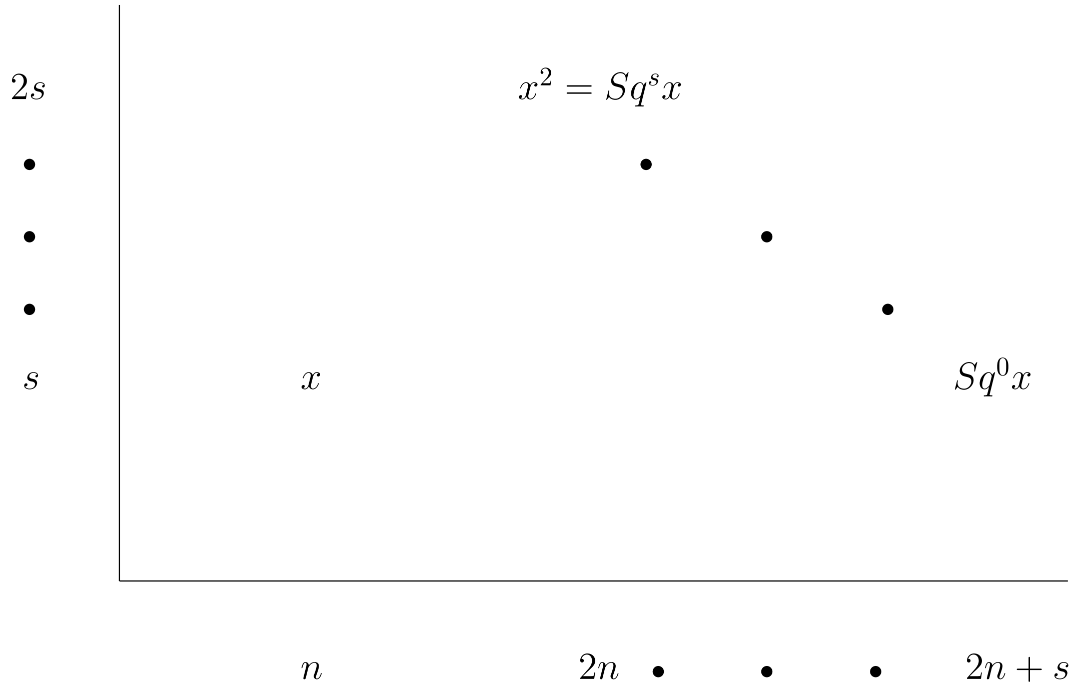


FIGURE 3.3. Steenrod operations in $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$

3.5. Families and the doomsday conjecture

The operation Sq^0 has many special properties. It is a ring homomorphism. It is induced by the squaring map of the dual Steenrod algebra. In particular, in the cobar resolution, it is given by the formula $Sq^0[x_1|\cdots|x_k] = [x_1^2|\cdots|x_k^2]$ [**26**]. Elements linked by Sq^0 share many properties. The Hopf invariant one elements h_i and the Kervaire invariant one elements h_i^2 are connected by Sq^0: $Sq^0(h_i) = h_{i+1}$ and $Sq^0(h_i^2) = h_{i+1}^2$.

DEFINITION 3.5.1. Given $a \in \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, let $a_0 = a$ and $a_{i+1} = Sq^0(a_i)$. We call the collection $\{a_i\}$ of elements of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ a *family*.

Elements of a family share many properties, with the exception of the first few, generally. For example, $d_2(h_{i+1}) = h_0 h_i^2$ if $i > 0$. For reasons which we will discuss in Section 5.5, the author made the following conjecture in the late 1970's. It has since been popularized and studied by Minami [**29, 30, 31**].

CONJECTURE 3.5.2 (The New Doomsday Conjecture). Only a finite number of nonzero elements in a family survive to E_∞ of the Adams spectral sequence.

It is called the *New* Doomsday Conjecture because it is a weaker version of the original Doomsday Conjecture, due to Joel Cohen, that there are only a finite number of nonzero elements in each filtration in E_∞ of the Adams spectral sequence for $\pi_* S$. That conjecture was probably based on the fact that only a finite number of Hopf invariant one elements survive to E_∞. The name 'Doomsday Conjecture' was applied partly because it would imply there are more differentials than anyone knows how to produce, and also because it would imply

Conjecture 3.5.3 (The Kervaire Invariant One Doomsday Conjecture). There are only a finite number of manifolds of Kervaire invariant one.

This follows from the Doomsday Conjecture because Browder showed ([**8**]) that a Kervaire invariant one manifold must have dimension $2(2^n-1)$ for some n and that its existence implies that h_n^2 is a permanent cycle in the mod 2 Adams spectral sequence $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_* S^0$. Since the h_n^2 are all in filtration 2, the Doomsday Conjecture implies that only a finite number of them can survive to E_∞, and hence only a finite number of Kervaire Invariant One manifolds can exist.

The Kervaire Invariant One Doomsday Conjecture was considered an unpleasant state of affairs, because the Kervaire invariant one elements are natural candidates for Hopf invariants of certain elements in the image of the J-homomorphism. In the late 1960's and early 1970's the non-existence of the Kervaire invariant one elements would have left the analysis of the Image of J in the EHP-sequence in an unclear state. Since then, our understanding of periodicity and the use of the Adams-Novikov spectral sequence to organize our knowledge of homotopy have sidestepped the need to solve the Kervaire invariant one problem. Mahowald [**24**] provides the definitive study of the Image of J; a nice summary can be found in Ravenel [**34**, pp. 39-44].

Mahowald definitively refuted the original Doomsday Conjecture when he showed that the filtration 2 elements $h_1 h_j$ all survive to E_∞ of the Adams spectral sequence to detect elements called $\eta_j \in \pi_{2^j} S$. However, this leaves the New Doomsday Conjecture intact, since the $h_1 h_j$ do not form a family: $Sq^0(h_1 h_j) = h_2 h_{j+1}$, $Sq^0(h_2 h_{j+1}) = h_3 h_{j+2}$, etcetera. With at most three exceptions, these elements are not permanent cycles (Mahowald and Tangora [**25**]).

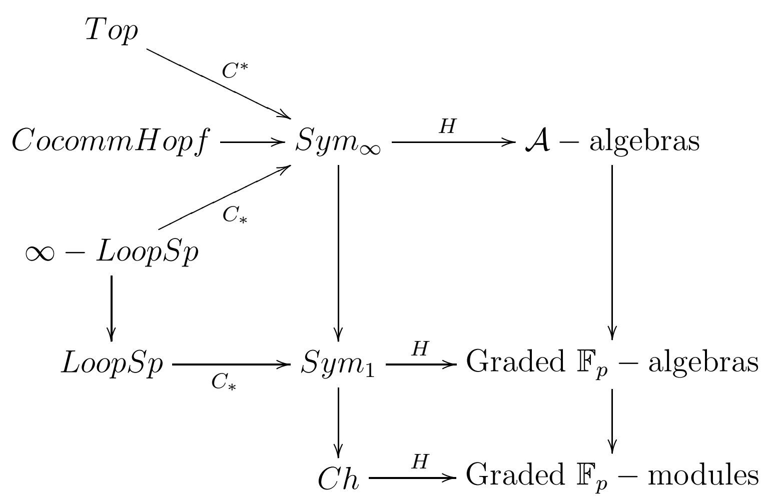
To summarize, the original Doomsday Conjecture is false, but the New Doomsday Conjecture and the Kervaire Invariant One Doomsday Conjecture remain open. As noted above, the elements $\{h_i^2\}$ are a family, so the New Doomsday Conjcture implies the Kervaire Invariant One Doomsday Conjecture. As of Spring 2000, it is known that h_n^2 survives to E_∞ if $n \leq 5$, and little is known about h_n^2 for $n > 5$ (see [**30**] for a survey of what is known). Despite the fact that our study of stable homotopy has largely sidestepped the need to resolve the Kervaire invariant one problem, it is a dramatic unresolved test problem, and the New Doomsday Conjecture seems a reasonable conjecture for which there is only a small amount of evidence either way.

CHAPTER 4

The General Algebraic Construction of Steenrod Operations

Peter May [**26**] has defined an algebraic category which carries the minimum structure needed to define generalized Steenrod operations, including the examples of the preceding chapter and many more. One can then define Steenrod operations in a particular setting by defining a functor to his category. In this chapter, we give a summary of the results in [**26**].

The following diagram shows the relation between the various categories and functors involved in the three examples of the preceding chapter. In it Sym_∞ denotes May's category, a category of chain complexes with additional structure to be defined precisely in section 4.3, Sym_1 denotes the category of chain complexes with a homotopy associative product, and Ch denotes the category of chain complexes. All these are to be taken over $\mathbb{F}_p$. Top is the category of topological spaces, $CocommHopf$ the category of cocommutative Hopf algebras (again over $\mathbb{F}_p$), and $\infty - LoopSp$ and $LoopSp$ are the categories of infinite loop spaces and one-fold loop spaces, respectively. The functors C_* and C^* are singular chains and cochains, and the functor H is homology. All the vertical functors are forgetful functors. We shall be concerned with Sym_∞; the remainder of the diagram is included simply to set the context for these results.



The reader may notice the absence of the category of spectra from this diagram. This is no accident. The cup product, which gives rise to the Steenrod operations, as in the preceding chapter, is an unstable operation, being derived from the diagonal map $\Delta : X \longrightarrow X \times X$ of spaces. Most spectra have no such coproduct. This is one of the special properties of suspension spectra. On the other hand, cohomology, is a stable functor: $H^*(\Sigma X) \cong H^*X$ (with a dimension shift, of course), and the stable cohomology operations such as the Steenrod operations do act on the cohomology of all spectra. Thus, it is natural, at least at our present level of understanding of spaces and spectra, that we use the category of topological spaces (or if we wish, simplicial complexes) in our construction of the Steenrod operations, even in the category of spectra.

Dave Benson, in [**7**], gives an alternative derivation of the Steenrod algebra for the cohomology of groups (which, by the Kan-Thurston theorem, is the same as the Steenrod algebra for the cohomology of spaces) from the Evens norm map. The structures we define here, which give rise to Steenrod operations and their properties, can be recognized in the symmetries implicit in his definition.

The algebra of operations which apply to all objects of Sym_∞ is a universal Steenrod algebra. When applied to objects of Sym_∞ coming from a particular category such as Top, $CocommHopf$, or $\infty - LoopSp$, additional relations hold. In particular, the chain complex of an infinite loop space is non-negatively graded, and hence bounded below, while the cochain complex of a topological space or cocommutative Hopf algebra, when thought of as a negatively graded chain complex (so that its differential will have degree -1), is bounded above. This gives the *Dyer-Lashof algebra*, the Steenrod algebra of operations which apply to the homology of infinite loop spaces, a very different character from the Steenrod algebras $\mathcal{A}$ and $\bar{\mathcal{A}}$ which apply to the cohomology of topological spaces and of cocommutative Hopf algebras, respectively. These latter two are closely related: setting Sq^0 (if $p = 2$) or $\mathcal{P}^0$ (if $p > 2$) to the identity in $\bar{\mathcal{A}}$ gives $\mathcal{A}$. Another closely related Steenrod algebra is the one which applies to the cohomology of simplicial restricted Lie algebras. It is the quotient of $\bar{\mathcal{A}}$ obtained by setting Sq^0 or $\mathcal{P}^0$ to zero. See May [**26**] for full details. Note that the presentation there was motivated by the application to n-fold loop spaces, $n \leq \infty$, and that there are additional operations and other subtleties which pertain when $n < \infty$. The relevant categories Sym_n sit between Sym_∞ and Sym_1 in the obvious way. For our applications, only $n = \infty$ is relevant.

Suppose that $\mathcal{C} \in Sym_\infty$. The Steenrod operations in $H(\mathcal{C})$ and their properties, will come from $H^*(\Sigma_k, \mathcal{C}^k)$, the cohomology of Σ_k with coefficients in the k^{th} tensor power of $\mathcal{C}$. Since $\mathcal{C}$ is an $\mathbb{F}_p$ vector space, we may restrict attention to the Sylow p-subgroup of Σ_k, which is the product of iterated wreath products of C_p's. The way in which the cohomology of wreath products can be expressed in terms of the cohomology of the factors leads to the fact that the indecomposable Steenrod operations come from the image of $H^*(\Sigma_p, \mathcal{C}^p)$ in $H^*(C_p, \mathcal{C}^p)$, while k-fold composites of Steenrod operations are governed by the cohomology of Σ_{p^k} and its image in the cohomology of its Sylow p-subgroup $C_p \wr C_p \wr \cdots \wr C_p$, the k-fold wreath product of C_p wth itself. A fundamental fact, and a remarkable one, is that all relations among Steenrod operations are generated by the quadratic relations. That is, all relations follow from the *Adem relations*, which come from the comparison between $C_p \wr C_p$ and Σ_{p^2}. The former essentially parameterizes two-fold composites before imposition of the Adem relations, while the latter takes these relations into account.

Since we are only summarizing the definition and properties of the Steenrod algebra(s) here, we will not do most of the homological calculations required. However, we need some small bit of information about the cohomology of the cyclic and symmetric groups to define the operations. This, we provide in the first section. The section that follows is an interesting aside, which follows immediately from a mild generalization of the calculation relevant to the symmetric group. It may safely be skipped by the reader who is interested only in the Steenrod operations. We then proceed to define Sym_∞ and the Steenrod operations.

4.1. The cohomology of cyclic and symmetric groups

Let C_k be the cyclic group of order k and Σ_k the symmetric group on k letters. The following proposition was proved in section 2.5.

THEOREM 4.1.1. *$H^*(C_2;\mathbb{F}_2) = \mathbb{F}_2[x]$ and $H^*(C_p;\mathbb{F}_p) = E[x]\otimes\mathbb{F}_p[y]$ if p is an odd prime, where $|x| = 1$ and $|y| = 2$.*

We will often write $y = x^2$ when $p = 2$, so that in all cases, $y \in H^2(C_p;\mathbb{F}_p)$ generates a polynomial subalgebra. Let us adopt the convention that subscripts indicate a cohomology class's degree: we will write x_i to mean that $x_i \in H^i$. The cyclic groups are our one exception to this convention; their generators are referred to so frequently that the formulas are more readable without subscripts. Let $i : C_p \longrightarrow \Sigma_p$ be the inclusion of a Sylow subgroup.

THEOREM 4.1.2. *If p is an odd prime then $H^*(\Sigma_p;\mathbb{F}_p) = E[x_{2p-3}] \otimes \mathbb{F}_p[y_{2p-2}]$ and $i^*(x_{2p-3}) = xy^{p-2}$ and $i^*(y_{2p-2}) = y^{p-1}$.*

Let $\mathbb{F}_p(-1)$ denote $\mathbb{F}_p$ with Σ_p action given by the sign representation: $\sigma\cdot x = \mathrm{sgn}(\sigma)x$. Similarly, $\mathbb{F}_p(+1)$ and $\mathbb{F}_p$ will denote $\mathbb{F}_p$ with the trivial Σ_p action. Obviously, $\mathbb{F}_p(k)\otimes\mathbb{F}_p(j) \cong \mathbb{F}_p(kj)$ if we give the tensor product the diagonal action.

THEOREM 4.1.3. *If p is an odd prime then $H^*(\Sigma_p;\mathbb{F}_p(-1))$ is a free module over $H^*(\Sigma_p;\mathbb{F}_p)$ on one generator u_{p-1} and $i^*(u_{p-1}) = y^{(p-1)/2}$.*

Proof: The first follows from a standard result in the cohomology of groups which identifies the mod p cohomology of a group with the *stable* elements in the cohomology of its Sylow p-subgroup, and can be found in Adem and Milgram [**5**, 6.6 and 6.8], Benson [**6**, 3.6.19, p. 68], Brown [**9**, 10.3] or Thomas [**35**, 3.2 and 3.4]. An explicit proof of both can be found in May[1.4]V168. □

4.2. Splittings and the nonabelian groups of order pq

The results of this section are included simply because they shed a little light on the cohomology calculations of the preceding section, and because the non-abelian groups of order pq are good test cases for conjectures. First, we need a result about the classifying space of the cyclic groups.

All these results will be deduced from our knowledge of the cohomology of cyclic groups by simple algebra from a general splitting result due to Fred Cohen [**15**], generalizing a technique used by Richard Holzsager [**17**].

NB: Look this up to make sure it is quoted correctly. In particular, Σ or Σ^n and H^* or H_*.

THEOREM 4.2.1 (Cohen). *Let $n > 0$. The following are equivalent:*

1. *$(\Sigma^n X)_{(p)} \simeq A_1 \vee \cdots A_k$ with $H_*A_i \cong M_i$,*
2. *There exist $f_i : (\Sigma^n X)_{(p)} \longrightarrow (\Sigma^n X)_{(p)}$ such that $f_{i*}H_*\Sigma^n X \cong M_i$, $H_*\Sigma^n X \cong \oplus M_i$, $f_{i*} : M_i \longrightarrow M_i$ is an isomorphism, and $f_{i*} : M_i \longrightarrow M_j$ is 0 if $i \neq j$.*

Proof: Certainly, (1) implies (2). For the reverse, let A_i be the mapping telescope

$$A_i = \mathrm{Tel}(\Sigma^n X \xrightarrow{f_i} \Sigma^n X \xrightarrow{f_i} \Sigma^n X \xrightarrow{f_i} \cdots).$$

Then we have inclusion maps $\Sigma^n X \longrightarrow A_i$, which induce the projection $H_*\Sigma^n X \longrightarrow M_i$. Since $\Sigma^n X$ is a suspension, we may add them to get a map $\Sigma^n X \longrightarrow A_1 \vee \cdots A_k$, which is a homology isomorphism, and hence an equivalence. $\square$

From this Cohen deduced a result of Mimura, Nishida and Toda [**?**].

THEOREM 4.2.2. *Suppose X is a connected H-space of finite type with H^*X or H_*X primitively generated. Then*

$$(\Sigma X)_{(p)} \simeq A_1 \vee \cdots \vee A_{p-1}$$

where H^*A_i or H_*A_i, respectively, is spanned by monomials in H^*X or H_*X, resp., (raised one degree by the suspension) of length congruent to i modulo $(p-1)$.

Proof [15]: Let $\theta_k : X \xrightarrow{\Delta} X^k \xrightarrow{\mu} X$, be the composite of the product and the diagonal. Let $\{u_i\}$ be the primitive generators. Then it follows that $\theta_{k*}(u_1 \cdots u_j) = k^j u_1 \cdots u_j$. Let k be a unit mod p and let M_t be spanned by the suspensions of the $u_{i_1} \cdots u_{i_j}$ where $k^j \equiv t \pmod{p}$. If $\nu_i = \Sigma(i) - \theta_k$, where $\Sigma(i)$ is the degree i map (using the suspension coordinate), then ν_i annihilates M_i and is an isomorphism on the other M_j. Thus, $f_i = \nu_1 \circ \cdots \circ \widehat{\nu_i} \circ \cdots \circ \nu_{p-1}$ is an isomorphism on M_i and annihilates the other M_j. The splitting then follows from the preceding theorem. $\square$

Using this, we split BC_p as follows.

THEOREM 4.2.3. $BC_p \simeq B_1 \vee B_2 \vee \cdots \vee B_{p-1}$ *where $H^*(B_i; \mathbb{F}_p)$ is nonzero only in degrees congruent to $2i$ and $2i-1$ modulo $2(p-1)$.*

Proof: The monomials of length i are y^i and xy^{i-1}, generating degrees $2i$ and $2i-1$ respectively. $\square$

Now we turn our attention to the non-abelian groups of order pq. If $q \mid p-1$ we may define a semidirect product $G_{q,p} = C_p \rtimes C_q$; the extension

$$C_p \vartriangleleft G_{q,p} \longrightarrow C_q$$

is given by an inclusion $C_q \rightarrowtail Aut(C_p) \cong C_{p-1}$. Note that we are assuming that p is prime, but q may be any divisor of $p-1$. The collection $\{G_{q,p}\}$ for fixed p is exactly the collection of subgroups intermediate between C_p and $N_{\Sigma_p}(C_p)$:

$$C_p = G_{1,p} \vartriangleleft G_{q,p} \vartriangleleft G_{p-1,p} = N_{\Sigma_p}(C_p).$$

Under inclusions the collection is isomorphic as a lattice to the lattice of divisors of $p-1$.

THEOREM 4.2.4. *Localized at p,*

$$BG_{q,p} \simeq B_q \vee B_{2q} \vee \cdots \vee B_{p-1}.$$

In particular, $BN_{\Sigma_p}(C_p)_{(p)} \simeq B_{p-1} \simeq (B\Sigma_p)_{(p)}$ is the dominant summand, which is always present in a wedge decomposition of BG if G has Sylow p-subgroup C_p.

Proof: The action of C_q on C_p is the k^{th} power map, where k is a primitive q^{th} root of 1 (mod p). This multiplies by k^i in degrees $2i$ and $2i-1$, so the C_q invariants are exactly the elements in degrees congruent to 0 and -1 (mod $2q$). The inclusion of the B_{iq} into BC_p followed by the natural map into $BG_{q,p}$ is therefore an equivalence. $\square$

4.3. Definition and properties

The category Sym_∞ will be the category of Adem and Cartan objects in a category $\mathcal{C}(C_p, \mathbb{F}_p)$, which we define now.

Let R be a commutative ring with 1. (It will usually be $\mathbb{F}_p$ or $\mathbb{Z}$.) Let $\pi \subset \Sigma_r$ be a subgroup, and let it act on r-fold tensor products of R-modules by permuting factors, and on tensor products of $R[\pi]$-modules diagonally. Let C_p be the cyclic group, $C_p = \langle T \mid T^p = 1 \rangle$, which we include in Σ_p by sending T to the cyclic permutation $(1\ 2\ \ldots\ p)$. Here p is a prime which is fixed throughout this section.

Let $\mathcal{V}$ be a $R[\Sigma_r]$-free resolution of R. If $\pi = C_p$, let $\mathcal{W}$ be the standard $R[C_p]$-free resolution of R, with each $\mathcal{W}_i$ free over $R[C_p]$ on one generator e_i, $i \geq 0$, with differential

$$d(e_{2i}) = (1-T)e_{2i-1} \quad \text{and} \quad d(e_{2i+1}) = (1 + T + \cdots T^{p-1})e_{2i},$$

and with augmentation $\epsilon(e_i) = \delta_{i0}$. Otherwise, let $\mathcal{W}$ be any $R[\pi]$-free resolution of R.

Definition 4.3.1. Let the category $\mathcal{C}(\pi, R)$ have

objects: (K, θ) such that K is a $\mathbb{Z}$–graded homotopy associative differential R-algebra, and $\theta : \mathcal{W} \otimes K^r \longrightarrow K$ is a morphism of $R[\pi]$-complexes, satisfying

(1) $\theta \mid \langle e_0 \rangle \otimes K^r$ is the r-fold iterated product associated in some fixed order, and

(2) θ is $R[\pi]$-homotopic to a composite

$$\mathcal{W} \otimes K^r \longrightarrow \mathcal{V} \otimes K^r \xrightarrow{\phi} K$$

for some $R[\Sigma_r]$-morphism ϕ.

morphisms: $K \xrightarrow{f} K'$, a morphism of R-complexes such that

$$\begin{array}{ccc} \mathcal{W} \otimes K^r & \xrightarrow{\theta} & K \\ {\scriptstyle 1\otimes f^r}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{W} \otimes K' & \xrightarrow{\theta'} & K' \end{array}$$

is $R[\pi]$-homotopy commutative.

The induced homomorphism

$$\theta_* : H(\mathcal{W} \underset{C_p}{\otimes} K^p) \longrightarrow H(K)$$

is all we need to define Steenrod operations in $H(K)$. Of course, they will not have many desirable properties without additional structure.

DEFINITION 4.3.2. Let $(K, \theta) \in \mathcal{C}(C_p, \mathbb{F}_p)$ and let $x \in H_q(K)$ be represented by $a \in K$. Define $D_i(x) = \theta_*(e_i \otimes a^p) \in H_{pq+i}(X)$. If $p = 2$, let $P_s : H_q(K) \longrightarrow H_{q+s}(K)$ be

$$P_s(x) \;=\; \begin{cases} 0 & s < q \\ D_{s-q}(x) & s \geq q \end{cases} \tag{7}$$

If $p > 2$, let $P_s : H_q(K) \longrightarrow H_{q+2s(p-1)}(K)$ be

$$P_s(x) \;=\; \begin{cases} 0 & 2s < q \\ (-1)^s \nu(q) D_{(2s-q)(p-1)}(x) & 2s \geq q \end{cases} \tag{8}$$

and $\beta P_s : H_q(K) \longrightarrow H_{q+2s(p-1)-1}(K)$ be

$$P_s(x) \;=\; \begin{cases} 0 & 2s \leq q \\ (-1)^s \nu(q) D_{(2s-q)(p-1)-1}(x) & 2s > q \end{cases} \tag{9}$$

where $\nu(2j + \epsilon) = (-1)^j (m!)^\epsilon$ if $\epsilon \in \{0, 1\}$ and $m = (p-1)/2$.

For cohomology, we let $K^i = K_{-i}$, $P^s(x) = P_{-s}(x)$, and $\beta P^s(x) = \beta P_{-s}(x)$. Then

$$\begin{array}{ll} P^s : H^q(K) \longrightarrow H^{q+s}(K) & p = 2 \\ \left.\begin{array}{l} P^s : H^q(K) \longrightarrow H^{q+2s(p-1)}(K) \\ \beta P^s : H^q(K) \longrightarrow H^{q+2s(p-1)+1}(K) \end{array}\right\} & p > 2 \end{array}$$

Clearly this is a mere change of notation which does not affect the results. We will state most of the results in the cohomological form, as this is the application we are interested in here. One other notational convention should be noted: for the application to loop spaces, the operation P_s is generally written Q_s and called a *Dyer-Lashof* operation.

The operation βP_s is not in general the composite of a Bockstein with P_s. For this to hold (see the following Proposition) we need to assume that (K, θ) is *reduced mod* p, i.e., $(K, \theta) = (\widetilde{K} \otimes \mathbb{F}_p, \widetilde{\theta} \otimes 1)$ for a flat $\widetilde{K}$, where $(\widetilde{K}, \widetilde{\theta}) \in \mathcal{C}(C_p, \mathbb{Z})$.

We say that (K, θ) is *unital* if K has a 2-sided homotopy identity $\eta : R \longrightarrow K$ in $\mathcal{C}(\pi, R)$, where R is regarded as an object in $\mathcal{C}(\pi, R)$ with $\theta = \epsilon \otimes 1 : \mathcal{W} \otimes R^r \longrightarrow R^r \cong R$. The element $e = \eta_*(1) \in H^0(K)$ is then the identity element for the product in $H^*(K)$.

THEOREM 4.3.3. *The operations P^s and βP^s are natural homomorphisms with the following properties.*

(1) *The P^s and βP^s account for all nonzero operations, in the sense that the other D_i are zero.*
(2) *If $x \in H^q(K)$ then*

$$\begin{array}{lll} (p = 2) & P^q(x) = x^2 & \\ & P^s(x) = 0 & s > q \\ (p > 2) & P^s(x) = x^p & 2s = q \\ & P^s(x) = 0 & 2s > q \\ & \beta P^s(x) = 0 & 2s \geq q \end{array}$$

(3) *If $p > 2$ and (K, θ) is reduced mod p then βP^s is the composite of the Bockstein and P^s. If $p = 2$ then composition with the Bockstein β satisfies $\beta P^{s-1} = s P^s$.*
(4) *If (K, θ) is unital then $P^s(e) = 0$ if $s \neq 0$ and $\beta P^s(e) = 0$ for all s.*

Proof: This is proved in [**26**, 2.3, 2.4 and 2.5]. Note that (1) follows from the way that the cohomology of the symmetric group restricts to its Sylow p-subgroup (4.1.2 and 4.1.3). $\square$

Define a *tensor product* for (K,θ) and (K',θ') in $\mathcal{C}(\pi,R)$ by $(K\otimes K',\widetilde{\theta})$, where $\widetilde{\theta}$ is the composite

$$\mathcal{W}\otimes(K\otimes K')^r \xrightarrow{\psi\otimes\sigma} \mathcal{W}\otimes\mathcal{W}\otimes K^r\otimes K'^r \xrightarrow{1\otimes\tau\otimes 1} \mathcal{W}\otimes K^r\otimes\mathcal{W}\otimes K'^r \xrightarrow{\theta\otimes\theta'} K\otimes K'$$

Here the diagonal map $\psi:\mathcal{W}\longrightarrow\mathcal{W}\otimes\mathcal{W}$ is any $R[\pi]$-homomorphism covering $R\cong R\otimes R$, $\sigma:(K\otimes K')^r\longrightarrow K^r\otimes K'^r$ is the shuffle permutation, and τ is the transposition.

DEFINITION 4.3.4. We say that (K,θ) is a *Cartan object* if the product $K\otimes K\longrightarrow K$ is a morphism in $\mathcal{C}(\pi,R)$.

THEOREM 4.3.5. *If (K,θ) and (L,θ') are objects of $\mathcal{C}(C_p,\mathbb{F}_p)$, $x\in H^q(K)$ and $y\in H^r(L)$, then the* external Cartan formulas hold*:*

$$P^s(x\otimes y)=\sum_{i+j=s}P^i(x)\otimes P^j(y)$$

and, if $p>2$ then

$$\beta P^s(x\otimes y)=\sum_{i+j=s}\beta P^i(x)\otimes P^j(y)+(-1)^qP^i(x)\otimes\beta P^j(y).$$

If (K,θ) is a Cartan object, then the internal Cartan formulas *hold:*

$$P^s(xy)=\sum_{i+j=s}P^i(x)P^j(y)$$

and, if $p>2$ then

$$\beta P^s(xy)=\sum_{i+j=s}\beta P^i(x)P^j(y)+(-1)^qP^i(x)\beta P^j(y).$$

Proof: Proposition 2.6 and Corollary 2.7 of [**26**]. $\square$

Our last piece of structure gives rise to the Adem relations. Let $\mathcal{Y}$ be an $R[\Sigma_{p^2}]$-free resolution of R. Recall that, $C_p\wr C_p$ is a Sylow p-subgroup of Σ_{p^2}, by letting the normal $C_p^p\ \triangleleft\ C_p\wr C_p$ act by cyclic permutations within p disjoint blocks of size p, and the quotient C_p act by cyclically permuting these blocks. With its evident $C_p\wr C_p$ action, $\mathcal{W}\otimes(\mathcal{W})^p$ is a $C_p\wr C_p$-free resolution of R. By restriction from Σ_{p^2} to $C_p\wr C_p$, $\mathcal{Y}$ is also, so there is a natural $C_p\wr C_p$ homomorphism $w:\mathcal{W}\otimes(\mathcal{W})^p\longrightarrow\mathcal{Y}$, unique up to $C_p\wr C_p$ chain homotopy, covering the identity map of R.

Definition 4.3.6. We say that $(K,\theta) \in \mathcal{C}(C_p, \mathbb{F}_p)$ is an *Adem object* if there exists a Σ_{p^2} equivariant homomorphism $\xi : \mathcal{Y} \otimes K^{p^2} \longrightarrow K$, such that the diagram

$$\begin{array}{ccccc} (\mathcal{W} \otimes \mathcal{W}^p) \otimes K^{p^2} & \xrightarrow{w\otimes 1} & \mathcal{Y} \otimes K^{p^2} & & \\ & & & \searrow^{\xi} & \\ \big\downarrow{\scriptstyle 1\otimes\sigma} & & & & K \\ & & & \nearrow_{\theta} & \\ \mathcal{W} \otimes (\mathcal{W} \otimes K^p)^p & \xrightarrow{1\otimes\theta^p} & \mathcal{W} \otimes K^p & & \end{array}$$

is $C_p \wr C_p$-equivariantly homotopy commutative, where σ is the evident shuffle homomorphism.

Theorem 4.3.7. *If (K_i, θ_i) are Adem objects, $i = 1,\ 2$, then $(K_1, \theta_1) \otimes (K_2, \theta_2)$ is an Adem object.*

In order to state the Adem relations in a uniform manner let us adopt the convention that $\beta^0 P^s = P^s$ and $\beta^1 P^s = \beta P^s$. In the cohomology of an object which is reduced mod p, these are true formulas, but in general they are merely notational conventions. If $p = 2$ then we require $\epsilon = 0$ when we write $\beta^\epsilon P^s$. Recall the 'sideways' notation for binomial coefficients: $(n, m) = (n + m)!/n!m!$.

Theorem 4.3.8. *The following relations hold on the cohomology of any Adem object.*

(1) *If $a < pb$ then*

$$\beta^\epsilon P^a P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} \beta^\epsilon P^{a+b-i} P^i$$

(2) *If $p > 2$ and $a \le pb$ then*

$$\begin{aligned} \beta^\epsilon P^a \beta P^b &= (1-\epsilon) \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} \beta P^{a+b-i} P^i \\ &\quad - \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi-1} \beta^\epsilon P^{a+b-i} \beta P^i \end{aligned}$$

Proof: This is Corollary 5.1 of [**26**], which is a reindexing for cohomology of Theorem 4.7 there. (Note that the first binomial coefficient in the second equation has a typographical error in [**26**] which we have corrected here.) □

For applications to topological spaces or simplicial sets, the necessary structure maps come from the diagonal maps $X \longrightarrow X^r$, which are perfectly symmetric in that they commute with all permutations of coordinates in X^r, together with the Alexander-Whitney maps, which relate the (co)chains of the product to the tensor product of the (co)chains. As noted in the previous chapter, this can only be symmetric up to chain homotopies, and these homotopies are neatly encoded in the homomorphisms $\theta : \mathcal{W} \otimes C^*(X)^r \longrightarrow C^*(X)$. This is worked out in detail in [**26**, Sections 7 and 8], where the following proposition is also shown to apply to any object whose coproduct is simply the diagonal map $D(x) = (x, x)$.

THEOREM 4.3.9. *In the cohomology of topological spaces or simplicial sets, P^0 is the identity operation.*

We reserve the notation $\mathcal{A}$ for this version of the Steenrod algebra. It is generated by β and the P^s, $s > 0$, modulo the ideal generated by the Adem relations. That these are all the relations follows from the action of $\mathcal{A}$ on the cohomology of the elementary abelian groups, as shown by Serre ([**?**]). The method of proof is straightforward. First note that the Adem relations allow one to express every element in terms of the admissable operations, those in which each successive pair P^aP^b satisfies $a \geq b$ and each successive pair $P^a\beta P^b$ satisfies $a > pb$. Then the Cartan formula allows one to easily calculate the action on $H^*(BC_p \times \cdots BC_p)$ and find a monomial for each admissable operation, on which it acts nontrivially, but all earlier monomials in lexicographic ordering act trivially. (CHECK ORDERING)

The algebra $\bar{\mathcal{A}}$ of operations which act on the cohomology of cocommutative Hopf algebras arises similarly from the coproduct of the Hopf algebra together with the Alexander-Whitney maps, purely by homological arguments. This was observed by Liulevicius ([**22**]). The details can be seen there or in [**26**] or [**12**, IV.2]. In contrast to the topological case,

$$P^0 : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s,pt}$$

so P^0 cannot act as the identity except in internal degree $t = 0$. In fact, Liulevicius shows that in the cobar construction,

$$P^0[a_1|\dots|a_k] = [a_1^p|\dots|a_k^p]$$

Applying this formula to the cohomology of the mod 2 Steenrod algebra, where $h_i = [\xi_1^{2^i}]$, we see immediately that

$$\begin{aligned} Sq^0 h_i &= h_{i+1} \\ Sq^1 h_i &= h_i^2, \text{ and} \\ Sq^j h_i &= 0 \text{ if } j > 1, \end{aligned}$$

the latter two following from the fact that $h_i \in \mathrm{Ext}^1$. (He also shows similar results at odd primes, and used them to solve the odd primary analog of the Hopf invariant one problem.)

REMARK 4.3.10. At first it appears that the Adem relations in $\bar{\mathcal{A}}$ are homogeneous: for example, instead of $Sq^1Sq^2 = Sq^3$, an inhomogeneous relation in the usual Steenrod algebra, we have $Sq^1Sq^2 = Sq^3Sq^0$ in $\bar{\mathcal{A}}$. However, this appearance is an illusion, because the algebras $\mathcal{A}$ and $\bar{\mathcal{A}}$ are augmented algebras, and generators for augmented algebras should lie in the augmentation ideal. Since Sq^0 maps to the identity operation in $\mathcal{A}$, it must have augmentation 1 for the quotient map $\bar{\mathcal{A}} \longrightarrow \mathcal{A}$ to be a map of augmented algebras. The appropriate generator of degree 0 and augmentation 0 is $\overline{Sq}^0 = Sq^0 - 1$, and the inhomogeneities then reappear: $Sq^1Sq^2 = Sq^3\overline{Sq}^0 + Sq^3$. This is relevant to attempts to use Priddy's method of Koszul resolutions to obtain information about $H^*\mathcal{A}$ from $H^*\bar{\mathcal{A}}$. If $\bar{\mathcal{A}}$ were in fact homogeneous, its cohomology would be easily computed. As it turns out, $H^*\bar{\mathcal{A}}$ is simply the tensor product of $H^*\mathcal{A}$ and an exterior algebra on 1 generator (traditionally called λ_{-1}) dual to Sq^0. See [**11**] for details.

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