\textbf{L}p \textbf{E}STIMATES FOR A \textbf{T}RILINEAR \textbf{P}SEUDO-\textbf{D}IFFERENTIAL OPERATOR WITH FLAG SYMBOLS

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\textbf{Abstract.} In this paper, we establish the L\textsuperscript{p} estimate for a trilinear pseudo-differential operator, where the symbol involved is given by the product of two standard symbols from the Hörmander class BS\textsubscript{0,0}. The study of this operator is motivated from the trilinear Fourier multiplier operator with flag singularities considered by C. Muscalu in [11].

\section{Introduction}

For \(n \geq 1\) we denote by \(\mathcal{M}(\mathbb{R}^n)\) the set of all bounded symbols \(m \in L^\infty(\mathbb{R}^n)\), smooth away from the origin and satisfying the classical Marcinkiewcz-Mikhlin-Hörmander condition
\[
|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{\alpha}}
\]
for every \(\xi \in \mathbb{R}^n \setminus \{0\} \) and sufficiently many multi-indices \(\alpha\). Denote by \(T_m\) by the \(n\)-linear operator
\[
T_m(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}} m(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i (\xi_1 + \cdots + \xi_n) \cdot x} d\xi,
\]
where \(\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\) and \(f_1, \ldots, f_n\) are Schwartz functions on \(\mathbb{R}\), denoted by \(\mathcal{S}(\mathbb{R})\). From the classical Coifman-Meyer theorem we know \(T_m\) extends to a bounded \(n\)-linear operator from \(L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_n}(\mathbb{R})\) to \(L^r(\mathbb{R})\) for \(1 < p_1, \ldots, p_n \leq \infty\) and \(1/p_1 + \cdots + 1/p_n = 1/r > 0\). In fact this property holds for the high dimensions when \(f_i \in L^{p_i}(\mathbb{R}^d), i = 1, \ldots, n\) and \(m \in \mathcal{M}(\mathbb{R}^{nd})\), see [4, 7, 9]. The case \(p \geq 1\) was proved by Coifman and Meyer [4] and was extended to \(p < 1\) by Grafakos and Torres [7] and Kenig and Stein [9]. Moreover, in the multiparameter setting, the same boundedness property is true, see [13, 12, 14], and also see [2] for a weaker restriction on the smoothness for the multiplier.

For the corresponding pseudo-differential variant of the classical Coifman-Meyer theorem, let the symbol \(\sigma(x, \xi)\) belong to the bilinear Hörmander symbol class \(BS_{1,0}^0\);
that is, \( \sigma \) satisfies the condition
\[
|\partial_x^l \partial_\xi^\alpha \sigma(x, \xi)| \lesssim \frac{1}{(1 + |\xi|)^{\alpha}}
\]
for any \( x \in \mathbb{R}, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) and sufficiently many indices \( l, \alpha \). We have the following

**Theorem 1.1.** The operator
\[
T_\sigma(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}^n} \sigma(x, \xi) \hat{f}_1(\xi) \cdots \hat{f}_n(\xi) e^{2\pi i (\xi_1 + \cdots + \xi_n) \cdot x} \, d\xi
\]
is bounded from \( L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_n}(\mathbb{R}) \) to \( L^r(\mathbb{R}) \) for \( 1 < p_1, \ldots, p_n \leq \infty \) and \( 1/p_1 + \cdots + 1/p_n = 1/r > 0 \), where \( f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}) \) and \( \sigma \) satisfies (1).

For the proof of the above theorem, see [1] for bilinear, high dimensional case and [12] for one dimensional, \( n \)-linear case. This boundedness property also holds in the multi-parameter setting, see [6]. Properties of multi-parameter and multilinear pseudo-differential operators of Coifman-Meyer type have also been studied in [8].

For the trilinear Coifman-Meyer type theorem, Muscalu [11] proved the following theorem where the multiplier involved is a product of two symbols and has flag singularities, that is, for \( m_1, m_2 \in \mathcal{M}(\mathbb{R}^2) \) satisfying
\[
|\partial_\xi^\alpha \partial_\eta^\beta m_1(\xi, \eta)| \lesssim \frac{1}{(|\xi| + |\eta|)^{\alpha + \beta}}
\]
\[
|\partial_\eta^\beta \partial_\zeta^\gamma m_2(\eta, \zeta)| \lesssim \frac{1}{(|\eta| + |\zeta|)^{\beta + \gamma}}
\]
for every \( \xi, \eta, \zeta \in \mathbb{R} \) and sufficiently many indices \( \alpha, \beta \) and \( \gamma \), we define
\[
T_{m_1, m_2}(f_1, f_2, f_3)(x) := \int_{\mathbb{R}^3} m_1(\xi, \eta) m_2(\eta, \zeta) \hat{f}_1(\xi) \hat{f}_2(\eta) \hat{f}_3(\zeta) e^{2\pi i (\xi + \eta + \zeta) \cdot x} \, d\xi d\eta d\zeta, \quad (4)
\]
where \( f_1, f_2, f_3 \in \mathcal{S}(\mathbb{R}) \). Then we have

**Theorem 1.2.** ([11]) The operator defined in (4) maps \( L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^r \) for \( 1 < p_1, p_2, p_3 \leq \infty \) with \( 1/p_1 + 1/p_2 + 1/p_3 = 1/r \) and \( 0 < r < \infty \). In addition, \( T_{m_1, m_2} \) also maps \( L^\infty \times L^p \times L^q \rightarrow L^s, L^p \times L^\infty \times L^t \rightarrow L^s, L^\infty \times L^t \times L^\infty \rightarrow L^t \) for every \( 1 < p, q, t < \infty \) and \( 1/p + 1/q = 1/s \).

Moreover, for the above theorem, the estimates like \( L^\infty \times L^\infty \times L^t \rightarrow L^t \) or \( L^\infty \times L^\infty \times L^\infty \rightarrow L^\infty \) are false, and these can be checked if we set \( f_2 \) to be identically 1.

Our main purpose is to consider a pseudo-differential operator corresponding to the above theorem, that is, let \( a(x, \xi, \eta), b(x, \eta, \zeta) \in BS^0_{1,0} \) be symbols satisfying the
conditions
\[
|\partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} a(x, \xi, \eta)| \lesssim \frac{1}{(1 + |\xi| + |\eta|)^{\alpha + \beta}}
\]
\[
|\partial_{\xi}^{\gamma} \partial_{\zeta}^{\delta} b(x, \eta, \zeta)| \lesssim \frac{1}{(1 + |\eta| + |\zeta|)^{\beta + \gamma}}
\]  
for every \(x, \xi, \eta, \zeta \in \mathbb{R}\) and sufficiently many indices \(\alpha, \beta, \gamma\), define the operator
\[
T_{ab}(f, g, h)(x) := \int_{\mathbb{R}^3} a(x, \xi, \eta) b(x, \eta, \zeta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i x (\xi + \eta + \zeta)} d\xi d\eta d\zeta.
\]  
(6)

It’s easy to see that the symbol \(a(x, \xi, \eta) \cdot b(x, \eta, \zeta)\) satisfies a less restrictive condition than the condition (1) for the symbol \(\sigma\) in Theorem 1.1. The main result of this paper is the following

**Theorem 1.3.** The operator \(T_{ab}\) defined as (6) is bounded from \(L^{p_1} \times L^{p_2} \times L^{p_3}\) to \(L^r\) for \(1 < p_1, p_2, p_3 < \infty\) with \(1/p_1 + 1/p_2 + 1/p_3 = 1/r\) and \(0 < r < \infty\). In addition, \(T_{ab}\) also maps \(L^\infty \times L^p \times L^q \rightarrow L^s\), \(L^p \times L^\infty \times L^q \rightarrow L^s\), \(L^\infty \times L^t \times L^\infty \rightarrow L^t\) for every \(1 < p, q, t < \infty\) and \(1/p + 1/q = 1/s\).

The proof of Theorem 1.3 is to reduce the trilinear pseudo-differential operator with the symbol of flag singularity to a localized version and takes advantage of the flag paraproducts from Muscalu’s work [11] on the \(L^p\) estimates for the Fourier multipliers with symbols of flag singularity. Namely, we need to prove an equivalent localized version Theorem 3.1 of Theorem 1.3 (see Muscalu and Schlag [12] for one-parameter case, and [6] for the multi-parameter setting). Moreover, the key to prove the localized result is that, conditions (5) allow us to only consider the dyadic intervals with lengths at most 1 in the flag paraproducts.

More precisely, in Section 3 we will show that our main theorem can be reduced to an estimate for a localized operator
\[
T_{ab}^{0,0}(f, g, h)(x) = \left( \int_{\mathbb{R}^3} a_0(\xi, \eta) b_0(\eta, \zeta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i x (\xi + \eta + \zeta)} d\xi d\eta d\zeta \right) \varphi_0(x),
\]
where \(\varphi_0(x)\) is a Schwartz function supported near the origin and \(a_0, b_0\) satisfy a stronger decay condition than the classical Hörmander-Mikhlin condition.

In Section 4, we will decompose the operator \(T_{ab}^{0,0}\) to some operators of different forms. Among these operators, some of them could be reduced to the classical pseudo-differential operator in Theorem 1.1, and the others could be written as flag
paraproducts, which are used in the proof of Theorem 1.2, in the forms of

\[ (T_1(f, g, h) \cdot \varphi_0)(x) = \sum_{J \in \mathcal{I}} \frac{1}{|J|} \langle f, \phi^1_J \rangle \langle B^1_J(g, h), \phi^2_J \rangle \phi^3_J \varphi_0 \]

where \[ B^1_J(g, h) = \sum_{\substack{J \in \mathcal{I} \setminus \mathcal{J} \cap \bigcup_{|\omega^j_J| \leq |\omega^2_J|} |\omega^j_J| \leq |\omega^2_J|}} \frac{1}{|J|} \langle g, \phi^1_J \rangle \langle h, \phi^2_J \rangle \phi^3_J, \]

but with dyadic intervals have lengths at most 1. Then by taking advantage of the flag paraproducts mentioned above, we will be able to prove the desired estimate for the localized version of our theorem in Section 5.

We end this introduction by briefly describing some recent works related to the results in this paper. In our recent paper [10], we study the bi-parameter pseudo-differential variant of Theorem 1.3. In order to study such bi-parameter pseudo-differential operator, a bi-parameter version of Theorem 1.2 has to be established first usually. However, such a result is hard to prove when the multipliers there have bi-parameter flag singularities involved. Fortunately, it turns out we can strengthen the conditions on the bi-parameter trilinear multipliers and get a Hölder type estimate for such strengthened bi-parameter trilinear multipliers. The \(L^p\) estimates for this bi-parameter trilinear multipliers will be sufficient in the study of bi-parameter trilinear pseudo-differential operators of flag symbols. That is, let \(m_3, m_4 \in \mathcal{B} \mathcal{M}_0(\mathbb{R}^{2n} \times \mathbb{R}^{2n})\) be smooth symbols that satisfy

\[
|\partial^\alpha_1 \partial^\alpha_2 \partial^\alpha_3 \partial^\alpha_4 m_3(\xi, \eta)| \lesssim \frac{1}{(1 + |\xi_1| + |\eta_1|)^{\alpha_1} (1 + |\xi_2| + |\eta_2|)^{\alpha_2}},
\]

\[
|\partial^\alpha_1 \partial^\alpha_2 \partial^\alpha_3 \partial^\alpha_4 m_4(\eta, \zeta)| \lesssim \frac{1}{(1 + |\eta_1| + |\xi_1|)^{\alpha_1} (1 + |\eta_2| + |\xi_2|)^{\alpha_2}},
\]

for every \(\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^n \times \mathbb{R}^n\) and all multi-indices \(\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)\) and \(\gamma = (\gamma_1, \gamma_2)\), then we can establish the following \(L^p\) estimates (see [10]):

**Theorem 1.4.** For \(f, g, h \in \mathcal{S}(\mathbb{R}^{2n})\), the bi-parameter operators

\[ T_{m_3, m_4}(f, g, h)(x) := \int_{\mathbb{R}^{6n}} m_3(\xi, \eta) m_4(\eta, \zeta) \tilde{f}(\xi) \tilde{g}(\eta) \tilde{h}(\zeta) e^{2\pi i (\xi + \eta + \zeta) \cdot x} \, d\xi d\eta d\zeta \]  

map \(L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^r\) for \(1 < p_1, p_2, p_3 < \infty\) with \(1/p_1 + 1/p_2 + 1/p_3 = 1/r\) and \(0 < r < \infty\).

Actually, from the proof of the above theorem, we can get a more general result without much difficulty. For \(l, n \geq 1\), let \(m(\xi) \in C^\infty(\mathbb{R}^l \times \mathbb{R}^n)\), where \(\xi = (\xi_i)_{i=1}^l\) and \(\xi_i = (\xi^1_i, \xi^2_i) \in \mathbb{R}^n \times \mathbb{R}^n\). We say \(m \in \mathcal{B} \mathcal{M}_0(\mathbb{R}^l \times \mathbb{R}^n)\) if

\[
|\partial^{\alpha_1, \alpha_2}_{\xi^{1}_i \xi^{1}_2} \cdot \partial^{\alpha_1, \alpha_2}_{\xi^{2}_i \xi^{2}_2} m(\xi)| \lesssim \frac{1}{(1 + |\xi_1| + \cdots + |\xi_l|)^{\alpha_1 + |\alpha_2|} (1 + |\xi^1_1| + \cdots + |\xi^1_l|)^{\alpha_1} + |\alpha_2|}.
\]
for every $\xi \in \mathbb{R}^{l \cdot 2n}$ and all multi-indices $\alpha, \alpha', \ldots, \alpha, \alpha'$. Then the following result has been proved in [10]:

**Theorem 1.5.** For integers $n, l \geq 1$, let

$$m(\xi) := \prod_{S \subseteq \{1, \ldots, l\}} m_S(\xi_S),$$

where $m_S \in B\mathcal{M}_0(\mathbb{R}^{\text{card}(S) \cdot 2n})$, the vector $\xi_S \in \mathbb{R}^{\text{card}(S) \cdot 2n}$ is defined by $\xi_S := (\xi_i)_{i \in S}$, where $\xi_i \in \mathbb{R}^{2n}$ and $\xi$ is the vector $\xi := (\xi_i)_{i=1}^l$. Every such symbol $m$ can define a $l$-linear operator

$$T_m(f_1, \ldots, f_l)(x) := \int_{\mathbb{R}^{2ln}} m(\xi) \hat{f}_1(\xi) \cdots \hat{f}_l(\xi) e^{2\pi i x(\xi_1 + \cdots + \xi_l)} \, d\xi,$$

where $f_1, \ldots, f_l$ are Schwartz functions on $\mathbb{R}^{2n}$. Then we have $T_m$ maps $L^{p_1} \times \cdots \times L^{p_l} \to L^p$ boundedly if $1 < p_1, \ldots, p_l < \infty$ and $1/p_1 + \cdots + 1/p_l = 1/p$.

Now we state the result for $L^p$ estimates for the corresponding bi-parameter trilinear pseudo-differential operators proved in [10]. Let

$$T_{ab}(f, g, h)(x) := \int_{\mathbb{R}^6} a(x, \xi, \eta) b(x, \eta, \zeta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i x(\xi_+ \eta + \zeta)} \, d\xi d\eta d\zeta,$$

where $f, g, h \in \mathcal{S}(\mathbb{R}^2)$, and the smooth symbols $a, b \in BBS_{1,0}^0$ satisfy the following conditions

$$|\partial^\alpha_x \partial^\beta_\eta \partial^\gamma_\zeta a(x, \xi, \eta)| \lesssim \frac{1}{(1 + |\xi_1| + |\eta_1|)^{\alpha_1} + |\beta_1|} \frac{1}{(1 + |\xi_2| + |\eta_2|)^{\alpha_2} + |\beta_2|};$$

$$|\partial^\alpha_x \partial^\beta_\eta \partial^\gamma_\zeta b(x, \eta, \zeta)| \lesssim \frac{1}{(1 + |\eta_1| + |\zeta_1|)^{\beta_1} + |\gamma_1|} \frac{1}{(1 + |\eta_2| + |\zeta_2|)^{\beta_2} + |\gamma_2|},$$

for every $x = (x_1, x_2), \xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R} \times \mathbb{R}$ and all multi-indices $l = (l_1, l_2), \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_1, \gamma_2)$. Our result established in [10] is the following

**Theorem 1.6.** The operators $T_{ab}$ defined as (9) map $L^{p_1} \times L^{p_2} \times L^{p_3} \to L^r$ for $1 < p_1, p_2, p_3 < \infty$ with $1/p_1 + 1/p_2 + 1/p_3 = 1/r$ and $0 < r < \infty$.

The main idea to prove Theorem 1.6 is to reduce the bi-parameter trilinear pseudo-differential operator to a localized version. Then by taking advantage of the $L^p$ estimates of the bi-parameter trilinear multipliers satisfying (7), we can establish Theorem 1.6. We refer to reader to [10] for more details.
2. NOTATIONS AND PRELIMINARIES

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing, $C^\infty$ functions in $\mathbb{R}$. Define the Fourier transform of a function $f$ in $\mathcal{S}(\mathbb{R})$ as

$$F(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} dx$$

extended in the usual way to the space of tempered distribution $\mathcal{S}'(\mathbb{R})$, which is the dual space of $\mathcal{S}(\mathbb{R})$.

Throughout the paper, we use $A \lesssim B$ to represent that there exists a universal constant $C > 1$ so that $A \leq CB$, and use the notation $A \asymp B$ to denote that $A \sim B$ and $B \sim A$.

We call the intervals in the form of $[2^k n, 2^k(n+1)]$ in $\mathbb{R}$ to be dyadic intervals, where $k, n \in \mathbb{Z}$. We denote by $\mathbb{D}$ the set of all such dyadic intervals.

**Definition 1.** For $I \in \mathbb{D}$, we define the approximate cutoff function as

$$\tilde{\chi}_I(x) := (1 + \frac{\text{dist}(x, I)}{|I|})^{-100} \quad (11)$$

**Definition 2.** Let $I \subseteq \mathbb{R}$ be an arbitrary interval. A smooth function $\varphi$ is said to be a bump adapted to $I$ if and only if one has

$$|\varphi^{(l)}| \leq C_l C_M \frac{1}{|I|^l} \frac{1}{(1 + |x - x_I|/|I|)^M}$$

for every integer $M \in \mathbb{N}$ and sufficiently many derivatives $l \in \mathbb{N}$, where $x_I$ denotes the center of $I$ and $|I|$ is the length of $I$.

If $\varphi_I$ is a bump adapted to $I$, we say that $|I|^{1/p} \varphi_I$ is an $L^p$-normalized bump adapted to $I$, for $1 \leq p \leq \infty$.

**Definition 3.** A sequence of $L^2$-normalized bumps $(\Phi_I)_{I \in \mathbb{D}}$ adapted to dyadic intervals $I \in \mathbb{D}$ is called a non-lacunary sequence if and only if for each $I \in \mathbb{D}$ there exists an interval $\omega_I = \omega_{|I|}$ symmetric with respect to the origin so that $\text{supp} \Phi_I \subseteq \omega_I$ and $|\omega_I| \sim |I|^{-1}$.

**Definition 4.** A sequence of $L^2$-normalized bumps $(\Phi_I)_{I \in \mathbb{D}}$ adapted to dyadic intervals $I \in \mathbb{D}$ is called a lacunary sequence if and only if for each $I \in \mathbb{D}$ there exists an interval $\omega_I = \omega_{|I|}$ so that $\text{supp} \Phi_I \subseteq \omega_I$, $|\omega_I| \sim |I|^{-1} \sim \text{dist}(0, \omega_I)$ and $0 \notin 5\omega_I$.

**Definition 5.** Let $\mathcal{I}, \mathcal{J} \subseteq \mathbb{D}$ be two families of dyadic intervals with lengths at most 1. Suppose that $(\phi^j_I)_{I \in \mathcal{I}}$ for $j = 1, 2, 3$ are three families of $L^2$-normalized bump functions such that the family $(\phi^3_I)_{I \in \mathcal{I}}$ is non-lacunary while the families $(\phi^j_I)_{I \in \mathcal{I}}$ for $j \neq 2$ are both lacunary, and $(\phi^j_I)_{I \in \mathcal{J}}$ for $j = 1, 2, 3$ are three families of $L^2$-normalized bump functions, where at least two of the three are lacunary.
We define as in [11] the discrete model operators $T_1$ and $T_{1,k_0}$ for a positive integer $k_0$ by

$$T_1(f, g, h) = \sum_{I \in \mathcal{I}} \frac{1}{|I|^2} \langle f, \phi_I^1 \rangle \langle B_I^1(g, h), \phi_I^3 \rangle$$  \hspace{1cm} (12)

where

$$B_I^1(g, h) = \sum_{J \in \mathcal{J}} \frac{1}{|J|^2} \langle g, \phi_J^1 \rangle \langle h, \phi_J^2 \rangle \phi_J^3$$  \hspace{1cm} (13)

$$T_{1,k_0}(f, g, h) = \sum_{I \in \mathcal{I}} \frac{1}{|I|^2} \langle f, \phi_I^1 \rangle \langle B_{I,k_0}^1(g, h), \phi_I^3 \rangle$$  \hspace{1cm} (14)

where

$$B_{I,k_0}^1(g, h) = \sum_{J \in \mathcal{J}} \frac{1}{|J|^2} \langle g, \phi_J^1 \rangle \langle h, \phi_J^2 \rangle \phi_J^3$$  \hspace{1cm} (15)

3. Reduction to a localized version

To prove the theorem, we proceed as follows. First pick a sequence of smooth functions $(\varphi_n)_n \in \mathbb{Z}$ such that $\text{supp} \varphi_n \subseteq [n-1, n+1]$ and

$$\sum_{n \in \mathbb{Z}} \varphi_n = 1.$$  

Then we can decompose the operator $T_{ab}$ in (6) as

$$T_{ab} = \sum_{n \in \mathbb{Z}} T_{ab}^n$$

where

$$T_{ab}^n(f, g, h)(x) := T_{ab}(f, g, h)(x) \varphi_n(x).$$

Suppose we can prove the estimate

$$\|T_{ab}^n(f, g, h)\|_r \lesssim \|f \tilde{X}_I\|_{p_1} \|g \tilde{X}_I\|_{p_2} \|h \tilde{X}_I\|_{p_3},$$  \hspace{1cm} (16)

where $I_n$ is the interval $[n, n+1]$, and $\tilde{X}_I$ is defined as in (11).

Then our main Theorem 1.3 can be proved by the following estimate

$$\|T_{ab}(f, g, h)\|_r \lesssim \left( \sum_{n \in \mathbb{Z}} \|T_{ab}^n(f, g, h)\|_r^{p_1} \right)^{1/p_1} \lesssim \left( \sum_{n \in \mathbb{Z}} \|f \tilde{X}_I\|_{p_1} \|g \tilde{X}_I\|_{p_2} \|h \tilde{X}_I\|_{p_3} \right)^{1/r}$$

$$\lesssim \left( \sum_{n \in \mathbb{Z}} \|f \tilde{X}_I\|_{p_1}^{p_1} \right)^{1/p_1} \left( \sum_{n \in \mathbb{Z}} \|g \tilde{X}_I\|_{p_2}^{p_2} \right)^{1/p_2} \left( \sum_{n \in \mathbb{Z}} \|h \tilde{X}_I\|_{p_3}^{p_3} \right)^{1/p_3}$$

$$\lesssim \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}.$$  

Thus, we only need to prove (16).
Consider that for a fixed $n_0 \in \mathbb{Z}$, we have
\[ T_{ab}^{n_0}(f, g, h)(x) = \int_{\mathbb{R}^3} a(x, \xi, \eta)\hat{\varphi}_{n_0}(x)b(x, \eta, \zeta)\varphi_{n_0}(x)\varphi_{n_0}(x)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i(x+\eta+\zeta)}d\xi d\eta d\zeta, \]
where $\varphi_{n_0}$ is a smooth function supported on the interval $[n_0 - 2, n_0 + 2]$ and equals 1 on the support of $\varphi_{n_0}$. Then we rewrite the symbols $a(x, \xi, \eta)\hat{\varphi}_{n_0}(x)$ and $b(x, \eta, \zeta)\hat{\varphi}_{n_0}(x)$ by using Fourier series with respect to the $x$ variable
\[ a(x, \xi, \eta)\hat{\varphi}_{n_0}(x) = \sum_{t_1 \in \mathbb{Z}} a_{t_1}(\xi, \eta)e^{2\pi i t_1 x}, \]
\[ b(x, \eta, \zeta)\hat{\varphi}_{n_0}(x) = \sum_{t_2 \in \mathbb{Z}} b_{t_2}(\xi, \eta)e^{2\pi i t_2 x}, \]
where by taking advantage of conditions (5) we can have
\[ |\partial_{\xi, \eta}^{\alpha, \beta} a_{t_1}(\xi, \eta)| \lesssim \frac{1}{(1 + |t_1|)^M (1 + |\xi| + |\eta|)^{\alpha + \beta}}, \]
\[ |\partial_{\eta, \zeta}^{\beta, \gamma} b_{t_2}(\eta, \zeta)| \lesssim \frac{1}{(1 + |t_2|)^M (1 + |\eta| + |\gamma|)^{\beta + \gamma}} \]
for a large number $M$ and sufficiently many indices $\alpha, \beta, \gamma$. Note the decay in $l_1, l_2$ means we only need to consider the case for $l_1, l_2 = 0$, which is given by
\[ (T_{ab}^{n_0,0,0}(f, g, h)(x) = (\int_{\mathbb{R}^3} a_0(\xi, \eta) b_0(\eta, \zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i(x+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_{n_0}(x), \]
where symbols $a_0, b_0$ satisfy the following conditions
\[ |\partial_{\xi, \eta}^{\alpha, \beta} a_0(\xi, \eta)| \lesssim \frac{1}{(1 + |\xi| + |\eta|)^{\alpha + \beta}}, \]
\[ |\partial_{\eta, \zeta}^{\beta, \gamma} b_0(\eta, \zeta)| \lesssim \frac{1}{(1 + |\eta| + |\gamma|)^{\beta + \gamma}}. \] (17)
Using the translation invariance, we only need to prove the following localized result for $n_0 = 0$

**Theorem 3.1.** The operator
\[ T_{ab}^{0,0}(f, g, h)(x) = (\int_{\mathbb{R}^3} a_0(\xi, \eta) b_0(\eta, \zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i(x+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x) \] (18)
has the following boundedness property
\[ \|T_{ab}^{0,0}(f, g, h)\|_r \lesssim \|f\hat{\chi}_{l_0}\|_{p_1}\|g\hat{\chi}_{l_0}\|_{p_2}\|h\hat{\chi}_{l_0}\|_{p_3} \] (19)
for $1 < p_1, p_2, p_3 < \infty$ and $1/p_1 + 1/p_2 + 1/p_3 = 1/r$, where $\varphi_0$ is a smooth function supported within $[-1, 1]$ and $a_0, b_0$ satisfy the conditions (17).

In addition, this estimate also holds for the cases where at most one $p_i = \infty$ for $i = 1, 2, 3$ or $p_1, p_3 = \infty, 1 < p_2 < \infty$. 

Now we are ready to do some decompositions to the operator in (18).

4. REDUCTION OF THE LOCALIZED OPERATOR

In this section, we will mainly show the problem can be reduced to some operators or paraproducts that we are familiar with.

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function such that $\text{supp } \hat{\varphi} \subset [-1, 1]$ and $\hat{\varphi}(\xi) = 1$ on $[-1/2, 1/2]$. Define $\psi \in \mathcal{S}(\mathbb{R})$ be the Schwartz function satisfying
$$\hat{\psi}(\xi) := \frac{\varphi(\xi/2) - \varphi(\xi)}{2},$$
and let
$$\hat{\psi}_k(\cdot) = \hat{\psi}(\cdot/2^k) \quad \text{and} \quad \hat{\psi}_{-1}(\cdot) = \hat{\varphi}(\cdot).$$

Note that
$$1 = \sum_{k \geq -1} \hat{\psi}_k, \quad \text{where supp } \hat{\psi} \subset [-2^{k+1}, -2^{k-1}] \cup [2^{k-1}, 2^{k+1}] \text{ for } k \geq 0.$$

Then for any $m, n \in \mathbb{Z}$, we use $m \gg n$ to denote $m - n > 100$ and $m \sim n$ to denote $|m - n| \leq 100$. Consider the decomposition
$$1(\xi, \eta, \zeta) = \left( \sum_{k_1' \geq -1} \sum_{k_1'' \geq -1} \hat{\psi}_{k_1'}(\xi) \hat{\psi}_{k_1''}(\eta) \right) \left( \sum_{k_2' \geq -1} \sum_{k_2'' \geq -1} \hat{\psi}_{k_2'}(\eta) \hat{\psi}_{k_2''}(\zeta) \right). \quad (20)$$

Without loss of generality, we consider
$$\left( \sum_{k_1' \geq -1} \sum_{k_1'' \geq -1} \hat{\psi}_{k_1'}(\xi) \hat{\psi}_{k_1''}(\eta) \right) = \sum_{k_1' > 100} \hat{\psi}_{k_1'}(\xi) \hat{\psi}_{k_1''}(\eta) + \sum_{-1 \leq k_1' \leq k_1''} \hat{\psi}_{k_1'}(\xi) \hat{\psi}_{k_1''}(\eta)$$
$$+ \sum_{k_1' \approx k_1''} \hat{\psi}_{k_1'}(\xi) \hat{\psi}_{k_1''}(\eta) \quad (21)$$
$$:= A + B + C + D,$$
where term D can be written out specifically, which contains finite number of terms:
$$D = \hat{\phi}(\xi)\hat{\varphi}(\eta) + \text{Others}$$

To estimate C, note in this case actually both $k_1'$ and $k_1''$ are at least 1. Suppose $k_1' > 100$, we have:
$$\sum_{k_1' > 100} \hat{\psi}_{k_1'}(\xi) \hat{\psi}_{k_1''}(\eta) = \sum_{k > 100} \hat{\psi}_k(\xi) \hat{\psi}_k(\eta)$$
and then
\[ C = \sum_{k>100} \hat{\psi}_k(\xi) \hat{\psi}_k(\eta) + \sum_{k>100} \hat{\psi}_k(\xi) \hat{\psi}_k(\eta) \]
where \( \text{supp} \hat{\psi}_k \subseteq [-2^{k+101}, -2^{k-101}] \cup [2^{k-101}, 2^{k+101}] \).

Estimates for \( A \) and \( B \) are quite similar:
\[ A = \sum_{k_1} \sum_{-1 \leq k \leq -100} \hat{\psi}_{k'}(\eta) \hat{\psi}_{k'}(\xi) = \sum_{k \geq 100} \hat{\psi}_k(\xi) \hat{\varphi}_k(\eta) \]
\[ B = \sum_{k_1} \sum_{-1 \leq k \leq -100} \hat{\varphi}_{k'}(\xi) \hat{\varphi}_{k'}(\eta) = \sum_{k \geq 100} \hat{\varphi}_k(\xi) \hat{\psi}_k(\eta), \]
where \( \varphi_k \) is a Schwartz function with \( \text{supp} \hat{\varphi}_k \subseteq [-2^{k-100}, 2^{k+100}] \). For \( k \geq 0 \) we call the families like \( (\psi_k)_k \) to be \( \Psi \) type functions, whose Fourier transform have almost disjoint supports for different scales and call the families like \( (\varphi_k)_k \) to be \( \Phi \) type functions, whose Fourier transforms have overlapping supports for different scales.

In the rest of this paper, for convenience purpose we don’t distinguish between \( \psi_k \) and \( \hat{\psi}_k \), since they are of the same type and have comparative scales for the supports of their Fourier transforms, and we always use \( \psi_k \) to represent such \( \Psi \) type functions. Similarly we always use \( \varphi_k \) to represent a \( \Phi \) type function. With such notations we can write (21) as
\[ (\sum_{k_1 \geq -1} \hat{\psi}_{k'}(\xi))(\sum_{k_1 \geq -1} \hat{\psi}_{k'}(\eta)) = \sum_{k \geq 100} \hat{\psi}_k(\xi) \hat{\varphi}_k(\eta) + \sum_{k \geq 100} \hat{\varphi}_k(\xi) \hat{\psi}_k(\eta) + \sum_{k \geq 100} \hat{\psi}_k(\xi) \hat{\psi}_k(\eta) + D. \]

Later from the proof, we will see in (24) the three summations work similarly, since what we really need is at least one lacunary family in each summation. And all the functions in \( D \) play a same role as \( \hat{\varphi}(\xi) \hat{\varphi}(\xi) \), which means we actually can replace (24) by an equivalently version, which is
\[ \sum_{k \geq 0} \hat{\phi}_k^1(\xi) \hat{\phi}_k^1(\eta) + \hat{\varphi}(\xi) \hat{\varphi}(\xi), \]
where at least one of the families \( (\hat{\phi}_k^1(\xi))_k \) and \( (\hat{\phi}_k^2(\xi))_k \) is \( \Psi \) type.
Now to deal with (20), it’s equivalent to consider

\[
1(\xi, \eta, \zeta) = \left( \sum_{k_1' \geq -1} \sum_{k_2' \geq -1} \widehat{\psi}_k'(\xi) \widehat{\psi}_k'(\eta) \right) \left( \sum_{k_2' \geq -1} \sum_{k_2'' \geq -1} \widehat{\psi}_{k_2'}(\eta) \widehat{\psi}_{k_2''}(\zeta) \right)
\]

where for convenience purpose the symbol \(\approx\) is used to show the equivalence, and we will simply treat \(1(\xi, \eta, \zeta) = E + F + G + H\) in the rest of the paper.

Then by using the above and (18), we can decompose the localized operator as

\[
T_{ab}^{0,0}(f, g, h)(x) = \left( \int_{\mathbb{R}^3} a_0(\xi, \eta) b_0(\eta, \zeta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i (\xi + \eta + \zeta)} d\xi d\eta d\zeta \right) \varphi_0(x)
\]

\[
= \left( \int_{\mathbb{R}^3} a_0(\xi, \eta) b_0(\eta, \zeta) \right) (E + F + G + H) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i (\xi + \eta + \zeta)} d\xi d\eta d\zeta \varphi_0(x)
\]

\[
:= T_{ab}^{E,0,0} + T_{ab}^{F,0,0} + T_{ab}^{G,0,0} + T_{ab}^{H,0,0}.
\]

4.1. Estimates for \(T_{ab}^{H,0,0}\).

Recall

\[
T_{ab}^{H,0,0}(f, g, h)(x) = \left( \int_{\mathbb{R}^3} a_0(\xi, \eta) b_0(\eta, \zeta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i (\xi + \eta + \zeta)} d\xi d\eta d\zeta \right) \varphi_0(x)
\]

where note that \(m_H(\xi, \eta, \zeta) := a_0(\xi, \eta) b_0(\eta, \zeta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta)\) satisfies the condition

\[
| \partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma m_H(\xi, \eta, \zeta) | \lesssim \frac{1}{(1 + |\xi| + |\eta| + |\zeta|)^{\alpha + \beta + \gamma}}
\]

for sufficiently many indices \(\alpha, \beta, \gamma\). Then our desired localized estimate follows from Theorem 1.1, since we find the operator \(T_{ab}^{H,0,0}\) is just the localized operator used in the proof of Theorem 1.1, see [12, 6].

4.2. Estimates for \(T_{ab}^{F,0,0} + T_{ab}^{G,0,0}\).

Recall

\[
F = \left( \sum_{k_1} \phi^1_{k_1}(\xi) \phi^2_{k_1}(\eta) \right) \hat{\varphi}(\eta) \hat{\varphi}(\zeta),
\]
where at least one of the families \((\hat{\phi}_{k_1}^1)_{k_1}\) and \((\hat{\phi}_{k_1}^2)_{k_1}\) is \(\Psi\) type.

When \((\hat{\phi}_{k_1}^2)_{k_1}\) is \(\Psi\) type, Note that to make \(\sum_{k_1} \hat{\phi}_{k_1}^2(\eta) \hat{\phi}(\eta) \neq 0\), \(k_1\) will have a upper bound for the summation, say \(k_1 \leq 100\). Then desired estimate under this situation can be done by using the same way as in \(T_{ab}^{H,0,0}\), since only finite number of terms are involved.

When \((\hat{\phi}_{k_1}^1)_{k_1}\) is \(\Phi\) type, we must have \((\hat{\phi}_{k_1}^1)_{k_1}\) is \(\Psi\) type. Recall

\[
T_{ab}^{F,0,0}(f, g, h)(x) = 
\left( \sum_{k_1} \int_{\mathbb{R}^3} a_0(\xi, \eta) \phi_{k_1}^1(\xi) \phi_{k_1}^2(\eta) b_0(\eta, \zeta) \hat{\phi}(\eta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i(x+\eta+\zeta)d\xi d\eta d\zeta} \phi_0(x), \right) \tag{28}
\]

then we can use Fourier series to write

\[
a_0(\xi, \eta) \phi_{k_1}^1(\xi) \phi_{k_1}^2(\eta) = \sum_{n_1, n_2 \in \mathbb{Z}} C_{n_1, n_2}^{k_1} e^{2\pi i n_1 \xi / 2^{k_1}} e^{2\pi i n_2 \eta / 2^{k_1}}, \tag{29}
\]

where the Fourier coefficients \(C_{n_1, n_2}^{k_1}\) are given by

\[
C_{n_1, n_2}^{k_1} = \frac{1}{2^{2k_1}} \int \mathbb{R}^2 a_0(\xi, \eta) \phi_{k_1}^1(\xi) \phi_{k_1}^2(\eta) e^{-2\pi i n_1 \xi / 2^{k_1}} e^{-2\pi i n_2 \eta / 2^{k_1}}.
\]

By the decay condition (17) and the advantage that \((\hat{\phi}_{k_1}^1)_{k_1}\) is \(\Psi\) type, we can get the following by integration by parts sufficiently many times

\[
|C_{n_1, n_2}^{k_1}| \lesssim \frac{1}{(1 + |n_1| + |n_2|)^M}.
\]

Note by the decay in \(n_1, n_2\) we only need to consider the case when \(n_1, n_2 = 0\), see [12] and the proof in section 5 for more details, and similar things can be done for \(b_0(\eta, \zeta) \hat{\phi}(\eta) \hat{\phi}(\zeta)\). Then, we can use Hölder’s inequality and take advantage the fact that \(\phi\) is a bump function adapted to \([-1, 1]\) to prove the localized result for (28), that is,

\[
\|\left( \sum_{k_1} \int_{\mathbb{R}^3} \hat{\phi}_{k_1}^1(\xi) \hat{\phi}_{k_1}^2(\eta) \hat{\phi}(\eta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i (x+\eta+\zeta)d\xi d\eta d\zeta} \phi_0(x) \right)\|_r
\approx \|\left( \sum_{k_1} \int_{\mathbb{R}^3} \hat{\phi}_{k_1}^1(\xi) \hat{\phi}(\eta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i (x+\eta+\zeta)d\xi d\eta d\zeta} \phi_0(x) \right)\|_r
= \|\left( \sum_{k_1} \phi_{k_1}^1 * f(x) \phi_0(x)(\phi * g)(x) \phi_0(x)(\phi * h)(x) \phi_0(x) \right)\|_r
\leq \|\left( \sum_{k_1} \phi_{k_1}^1 * f(x) \phi_0(x) \right)\|_{p_1} \|\left( \phi * g)(x) \phi_0(x) \right)\|_{p_2} \|\left( \phi * h)(x) \phi_0(x) \right)\|_{p_3}
\leq \|f \tilde{\chi}_0\|_{p_1} \|g \tilde{\chi}_0\|_{p_2} \|h \tilde{\chi}_0\|_{p_3},
\]
where we take $\tilde{\phi}_0$ to be 1 on supp $\phi_0$ and supported in a slightly larger interval containing supp $\phi_0$. The last inequality is true since $(\varphi_{k_1})_{k_1}$ is $\Psi$ type. Also, in the above we can simply write $\sum_{k_1} \tilde{\phi}_{k_1}^2(\eta)\tilde{\varphi}(\eta) = \tilde{\varphi}(\eta)$ in the above since $k_1$ is positive.

4.3. Estimates for $T_{ab}^{E,0,0}$.

Recall

$$E = \left( \sum_{k_1 \geq 0} \phi_{k_1}^1(\xi)\phi_{k_1}^2(\eta) \right) \left( \sum_{k_2 \geq 0} \phi_{k_2}^1(\eta)\phi_{k_2}^2(\zeta) \right),$$

where at least one of the families $(\phi_{k_1}^1)_{k_1}$ and $(\phi_{k_1}^2)_{k_1}$ is $\Psi$ type and at least one of the families $(\phi_{k_2}^1)_{k_2}$ and $(\phi_{k_2}^2)_{k_2}$ is $\Psi$ type.

Also we consider the corresponding localized operator

$$T_{ab}^{E,0,0}(f,g,h)(x) = \left( \int_{\mathbb{R}^3} \left( \sum_{k_1} \phi_{k_1}^1(\xi)\phi_{k_1}^2(\eta) \right) a_0(\xi,\eta) \left( \sum_{k_2} \phi_{k_2}^1(\eta)\phi_{k_2}^2(\zeta) b_0(\eta,\zeta) \right) \tilde{f}(\xi)\tilde{g}(\eta)\tilde{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}d\xi d\eta d\zeta \varphi_0(x) \right).$$

By using Fourier series as before, we only need to consider the following operator

$$\left( \int_{\mathbb{R}^3} \left( \sum_{k_1} \phi_{k_1}^1(\xi)\phi_{k_1}^2(\eta) \right) \left( \sum_{k_2} \phi_{k_2}^1(\eta)\phi_{k_2}^2(\zeta) \right) \tilde{f}(\xi)\tilde{g}(\eta)\tilde{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}d\xi d\eta d\zeta \varphi_0(x) \right).$$

As usual we consider three cases of $E$

$$E = \left( \sum_{k_1 \geq k_2} + \sum_{k_1 < k_2} + \sum_{k_1 = k_2} \right) \left( \phi_{k_1}^1(\xi)\phi_{k_1}^2(\eta) \right) \left( \phi_{k_2}^1(\eta)\phi_{k_2}^2(\zeta) \right)$$

$$:= I + J + K,$$

and decompose

$$T_{ab}^{E,0,0} := T_{ab}^{I,0,0} + T_{ab}^{J,0,0} + T_{ab}^{K,0,0}.$$ Note $K$ is actually a symbol in $BS_{1,0}^0$, since $k$ is positive. That is,

$$T_{ab}^{K,0,0}(f,g,h)(x) = \left( \int_{\mathbb{R}^3} m_{K}(\xi,\eta,\zeta) \tilde{f}(\xi)\tilde{g}(\eta)\tilde{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}d\xi d\eta d\zeta \varphi_0(x) \right),$$

where $m_{K}(\xi,\eta,\zeta)$ satisfies the condition as (17). Thus, the desired localized estimate follows from the proof of Theorem 1.1, just as $T_{ab}^{H,0,0}$.

$T_{ab}^{I,0,0}$ and $T_{ab}^{J,0,0}$ are similar, we define $T_{ab}^{l}$ by the following equality

$$T_{ab}^{l}(f,g,h)(x) \cdot \varphi_0(x) =: T_{ab}^{l,0,0}(f,g,h)(x) = \left( \int_{\mathbb{R}^3} \left( \sum_{k_1} \phi_{k_1}^1(\xi)\phi_{k_1}^2(\eta) \right) \left( \sum_{k_2} \phi_{k_2}^1(\eta)\phi_{k_2}^2(\zeta) \right) \tilde{f}(\xi)\tilde{g}(\eta)\tilde{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}d\xi d\eta d\zeta \varphi_0(x) \right).$$

(30)
From [12, 11], we know $T\text{'}_{ab}$ can be written by using paraproducts, which is the following lemma.

**Lemma 4.1.** Define $T\text{'}_{ab}$ as in (30), then we can write

$$T\text{'}_{ab}(f, g, h)(x) = T_1(f, g, h)(x) + \sum_{l=1}^{M-1} \sum_{k_0=100}^{\infty} (2^{-k_0})^l T_{l, k_0}(f, g, h)(x) + \sum_{k_0=100}^{\infty} (2^{-k_0})^M T_{M, k_0}(f, g, h)(x)$$

where

$$T_1(f, g, h) = \sum_{J \in \mathcal{J}} \frac{1}{|J|^\frac{1}{2}} \langle f, \phi_J^1 \rangle \langle B_1^1(g, h), \phi_J^3 \rangle$$

with

$$B_1^1(g, h) = \sum_{J \in \mathcal{J}} \frac{1}{|J|^\frac{1}{2}} \langle g, \phi_J^1 \rangle \langle h, \phi_J^2 \rangle \phi_J^3$$

$$T_{l, k_0}(f, g, h) = \sum_{J \in \mathcal{J}} \frac{1}{|J|^\frac{1}{2}} \langle f, \phi_J^1 \rangle \langle B_{l, k_0}^1(g, h), \phi_J^3 \rangle$$

with

$$B_{l, k_0}^1(g, h) = \sum_{J \in \mathcal{J}} \frac{1}{|J|^\frac{1}{2}} \langle g, \phi_J^1 \rangle \langle h, \phi_J^2 \rangle \phi_J^3$$

In the above,

1. $T_1(f, g, h)$ and $B_1^1(g, h)$ are defined as (12) and (13) in definition (5).
2. For each $l$, $T_l(f, g, h)$ and $B_l^1(g, h)$ are of the type (14) and (15) in definition 5. $l$ here is actually involved in the families $(\phi_J^2)_l$ and $(\phi_J^3)_l$, but it won’t affect our proof since it does not change the types of those functions.
3. $M$ is a large positive integer, and the multiplier $m_{M, k_0}(\xi, \eta, \zeta)$ in $T_{M, k_0}$ satisfies the condition

$$|D_\xi^\alpha D_\eta^\beta D_\zeta^\gamma m_{M, k_0}(\xi, \eta, \zeta)| \lesssim (2^{k_0})^{\alpha+\beta+\gamma} \frac{1}{(1 + |\xi| + |\eta| + |\zeta|)^{\alpha+\beta+\gamma}}$$

for sufficiently many indices $\alpha, \beta, \gamma$.
4. All the dyadic intervals in $T_1$ and $T_{l, k_0}$ have lengths at most 1 for all $k_0 \geq 100, 1 \leq l \leq M - 1$.

**Proof.** We follow closely the work [11], where the Fourier expansions of $\hat{\phi}_{k_1}^1(\eta)$ are used to get the desired forms of paraproducts. The only two statements we need to show are that all the dyadic intervals there have lengths at most one and the decay number 1 in the denominator from (31). Actually both of them follow from the fact $k_1, k_2 \geq 0$.

So far we have reduced Theorem 3.1 to the estimate of the operator $T_{ab}^{1,0,0}$.
5. proof of Theorem 3.1

In this section by using the decomposition in Lemma 4.1, we are able to prove the localized estimate for $T_{ab}^{f,0,0}$, which will complete the proof of Theorem 3.1.

5.1. Estimates for $\sum_{k_0=100}^{\infty} (2^{-k_0})^M T_{M,k_0}(f,g,h)(x)$.

For this part, note that the condition (31) is almost the classical case. Then by repeating the work in [12, 6] we will see this condition can provide an estimate

$$\|T_{M,k_0}(f,g,h)\varphi_0(x)\| \lesssim C^2 10^{k_0} \|f \tilde{\chi}_{I_0}\|_{p_1} \|g \tilde{\chi}_{I_0}\|_{p_2} \|h \tilde{\chi}_{I_0}\|_{p_3}$$

which is accepted since we can choose $M$ large enough.

5.2. Estimates for $T_1(f,g,h)(x)$.

Taking advantage of that $|I| \leq 1$, we can split

$$T_1(f,g,h)(x) = \sum_{|I| \leq 5I_0} \frac{1}{|I|^2} \langle f, \phi_I^1 \rangle \langle B_I^1(g,h), \phi_I^2 \rangle \phi_I^3 + \sum_{|I| \leq (5I_0)^c} \frac{1}{|I|^2} \langle f, \phi_I^1 \rangle \langle B_I^1(g,h), \phi_I^2 \rangle \phi_I^3$$

$$= I + II. \tag{32}$$

For Part I, we do the following decompositions first

$$f = \sum_{n_1} f \chi_{I_{n_1}}, \quad g = \sum_{n_2} g \chi_{I_{n_2}}, \quad h = \sum_{n_3} h \chi_{I_{n_3}};$$

where $I_{n_i} = [n_i, n_i + 1], i = 1, 2, 3, n_i \in \mathbb{Z}$. Then we can write

$$T_1(f,g,h)(x) = \sum_{n_1} \sum_{n_2} \sum_{n_3} T_1(f \chi_{I_{n_1}}, g \chi_{I_{n_2}}, h \chi_{I_{n_3}})(x).$$

When $|n_1|, |n_2|, |n_3| \leq 10$, the desired estimate follows from Theorem 1.2

$$\| \sum_{|n_1| \leq 10} \sum_{|n_2| \leq 10} \sum_{|n_3| \leq 10} T_1(f \chi_{I_{n_1}}, g \chi_{I_{n_2}}, h \chi_{I_{n_3}})(x) \cdot \varphi_0(x) \|_r$$

$$\lesssim \| f \chi_{I_{n_1}} \|_{p_1} \| g \chi_{I_{n_2}} \|_{p_2} \| h \chi_{I_{n_3}} \|_{p_3}$$

$$\lesssim \| f \tilde{\chi}_{I_0} \|_{p_1} \| g \tilde{\chi}_{I_0} \|_{p_2} \| h \tilde{\chi}_{I_0} \|_{p_3},$$

where the last inequality holds from $\chi_{[-11,11]} \lesssim \tilde{\chi}_{I_0}(x)$. 
When \(|n_1|, |n_2|, |n_3| > 10\), we write
\[
\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x) \cdot \varphi_0(x)\|_r
\]
\[
= \left\| \sum_{I \in \mathcal{I}} \sum_{J \in \mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}} \frac{1}{|J|^{\frac{1}{2}}} \langle f\chi_{I_{n_1}}, \phi_I^1 \rangle \langle g\chi_{I_{n_2}}, \phi_I^1 \rangle \langle h\chi_{I_{n_3}}, \phi_I^3 \rangle \langle \phi_I^2, \phi_I^3 \rangle \phi_I^3(x) \varphi_0(x) \rangle_r
\]

Then we use Hölder’s inequality to get
\[
\left\| \frac{1}{|I|^{\frac{1}{2}}} \frac{1}{|J|^{\frac{1}{2}}} \langle f\chi_{I_{n_1}}, \phi_I^1 \rangle \langle g\chi_{I_{n_2}}, \phi_I^1 \rangle \langle h\chi_{I_{n_3}}, \phi_I^3 \rangle \langle \phi_I^2, \phi_I^3 \rangle \phi_I^3(x) \varphi_0(x) \rangle_r
\]
\[
\lesssim \frac{1}{|I|^{\frac{1}{2}}} \frac{1}{|J|^{\frac{1}{2}}} (1 + \frac{\text{dist}(I_{n_1}, J)}{|I|})^{M_1} (\|f\chi_{I_{n_1}}\|_{p_1} |I|^{\frac{p_1-1}{p_1}}) (1 + \frac{\text{dist}(I_{n_2}, J)}{|J|})^{N_1}
\]
\[
\cdot \int_{\mathbb{R}} \left(1 + \frac{\text{dist}(x, I)}{|I|} \right)^{-M_2} \left(1 + \frac{\text{dist}(x, J)}{|J|} \right)^{-N_2} dx
\]
\[
\lesssim \frac{1}{|I|} \frac{1}{|J|} \left( \frac{|I|}{|J|} \right)^{\frac{p_2-1}{p_2}} \left(1 + \frac{\text{dist}(I_{n_1}, J)}{|I|} \right)^{-M_1} \left(1 + \frac{\text{dist}(I_{n_2}, J)}{|J|} \right)^{-N_1} \left(1 + \frac{\text{dist}(I_{n_3}, J)}{|J|} \right)^{-N_2}
\]
\[
\cdot \int_{\mathbb{R}} \left(1 + \frac{\text{dist}(x, I)}{|I|} \right)^{-M_2} \left(1 + \frac{\text{dist}(x, J)}{|J|} \right)^{-N_2} dx \|f\chi_{I_{n_1}}\|_{p_1} \|g\chi_{I_{n_2}}\|_{p_2} \|h\chi_{I_{n_3}}\|_{p_3} (33)
\]

where \(M_j, N_j\) are sufficiently large integers and \(\phi_I^j, \phi_I'\) are \(L^2\)-normalized bump functions adapted to \(I, J\) for \(j = 1, 2, 3\).

We first consider the case when \(\text{dist}(I, J) \leq 3\). Recall we have the restriction that \(|\omega_j^2| \leq |\omega_j^3|\), which implies that \(|I|/|J| \leq 1\). By using the subadditivity of \(\| \cdot \|_r\) we have
\[
\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x) \cdot \varphi_0(x)\|_r
\]
\[
\lesssim \sum_{i,j \geq 0} \sum_{I \subseteq 5I_0, J \subseteq 9I_0} \left( \frac{|I|}{|J|} \right)^{(1 + \text{dist}(I_{n_1}, J))^{-M_1} (1 + \frac{\text{dist}(I_{n_2}, J)}{|J|})^{-N_1} (1 + \frac{\text{dist}(I_{n_3}, J)}{|J|})^{-N_2}}
\]
\[
\cdot \int_{\mathbb{R}} \left(1 + \frac{\text{dist}(x, I)}{|I|} \right)^{-M_2} \left(1 + \frac{\text{dist}(x, J)}{|J|} \right)^{-N_2} dx \|f\chi_{I_{n_1}}\|_{p_1} \|g\chi_{I_{n_2}}\|_{p_2} \|h\chi_{I_{n_3}}\|_{p_3} (33)
\]
\[
\lesssim \sum_{i,j \geq 0} \sum_{I \subseteq 5I_0, J \subseteq 9I_0} \left(2^i (1 + 2^i (|n_1| - 6))^{-M_1} (1 + 2^i (|n_2| - 9))^{-N_1} (1 + 2^i (|n_3| - 9))^{-N_2}
\]
\[
\cdot \left( \|f\chi_{I_{n_1}}\|_{p_1} \|g\chi_{I_{n_2}}\|_{p_2} \|h\chi_{I_{n_3}}\|_{p_3} \right)^r
\]
\[
\lesssim \left( (|n_1| - 6)^{-M_1} (|n_2| - 9)^{-N_1} (|n_3| - 9)^{-N_2} \right) \|f\chi_{I_{n_1}}\|_{p_1} \|g\chi_{I_{n_2}}\|_{p_2} \|h\chi_{I_{n_3}}\|_{p_3} (33).\]
Observe that for large enough integers $M_1, N_1, N_2$ we have

$$\chi_{I_{n_4}}(|n_1| - 6)^{-\frac{M_1}{2}} \lesssim \tilde{\chi}_{I_0}, \ \chi_{I_{n_2}}(|n_2| - 9)^{-\frac{N_1}{2}} \lesssim \tilde{\chi}_{I_0}, \ \chi_{I_{n_3}}(|n_3| - 9)^{-\frac{N_2}{2}} \lesssim \tilde{\chi}_{I_0}.$$ 

Thus,

$$\left| \sum_{|n_1| > 10} \sum_{|n_2| > 10} \sum_{|n_3| > 10} T_1(f \chi_{I_{n_1}}, g \chi_{I_{n_2}}, h \chi_{I_{n_3}})(x) \cdot \varphi_0(x) \right|_r ^ r \lesssim \left| \sum_{|n_1| > 10} \sum_{|n_2| > 10} \sum_{|n_3| > 10} \left( (|n_1| - 6)^{-M_1} (|n_2| - 9)^{-N_1} (|n_3| - 9)^{-N_2} \cdot \|f \chi_{I_{n_1}}\|_{p_1} \|g \chi_{I_{n_2}}\|_{p_2} \|h \chi_{I_{n_3}}\|_{p_3} \right) \right|_r ^ r \lesssim \left( \sum_{|n_1| > 10} \sum_{|n_2| > 10} \sum_{|n_3| > 10} \left( (|n_1| - 6)^{-M_1} (|n_2| - 9)^{-N_1} (|n_3| - 9)^{-N_2} \right) \cdot \|f \tilde{\chi}_{I_0}\|_{p_1} \|g \tilde{\chi}_{I_0}\|_{p_2} \|h \tilde{\chi}_{I_0}\|_{p_3} \right) ^ r.$$ 

For the other possibility, that is, when dist$(I, J) > 3$, we consider whether $J$ is close to $I_{n_2}$ or $I_{n_3}$. Without loss of generality, we assume dist$(J, I_{n_2}) \leq 2$, dist$(J, I_{n_3}) > 2$, and other cases will follow in the similar way. Using the notation $J_m = [m, m + 1], m \in \mathbb{Z}$ and (33) we can get

$$\|T_1(f \chi_{I_{n_1}}, g \chi_{I_{n_2}}, h \chi_{I_{n_3}})(x) \cdot \varphi_0(x)\|_r ^ r \lesssim \sum_{i,j \geq 0} \sum_{|I| = 2^{-i}} \sum_{|J| = 2^{-j}} \left\{ \left( \frac{1}{|I|} \left( 1 + \frac{\text{dist}(I_{n_1}, I)}{|I|} \right)^{-M_1} \left( 1 + \frac{\text{dist}(I_{n_2}, J)}{|J|} \right)^{-N_1} \right) \cdot \left( \frac{1}{|J|} \left( 1 + \frac{\text{dist}(I_{n_3}, J)}{|J|} \right)^{-N_2} \cdot \int_R \left( 1 + \frac{\text{dist}(x, I)}{|I|} \right)^{-M_2} \left( 1 + \frac{\text{dist}(x, J)}{|J|} \right)^{-N_3} dx \right) \cdot \|f \chi_{I_{n_1}}\|_{p_1} \|g \chi_{I_{n_2}}\|_{p_2} \|h \chi_{I_{n_3}}\|_{p_3} \right\} ^ r.$$ 

$$\lesssim \sum_{i,j \geq 0} \sum_{|I| = 2^{-i}} \sum_{|J| = 2^{-j}} \left\{ \left( 2^i \left( 1 + 2^i (|n_1| - 6)^{-M_1} \right) \left( 1 + 2^i (|m_3| - 6)^{-M_1} \right) \right)^{-N_1} \cdot \left( 2^j \left( 1 + 2^j (|n_2| - 9)^{-N_1} \right) \left( 1 + 2^j (|m_3| - 9)^{-N_1} \right) \right)^{-N_2} \cdot \|f \chi_{I_{n_1}}\|_{p_1} \|g \chi_{I_{n_2}}\|_{p_2} \|h \chi_{I_{n_3}}\|_{p_3} \right\} ^ r.$$ 

where $N_0 = \min\{M_2, N_3\}$ is sufficiently large and we use $m \sim n_2$. 


Now we take the sum over $n_1, n_2, n_3$ and get
\[
\left\| \sum_{|n_1|>10} \sum_{|n_2|>10} \sum_{|n_3|>10} T_1(f \chi_{I_{n_1}}, g \chi_{I_{n_2}}, h \chi_{I_{n_3}})(x) \cdot \varphi_0(x) \right\|_r \\lesssim \sum_{|n_1|>10} \sum_{|n_2|>10} \sum_{|n_3|>10} (|n_1| - 6)^{-M_1} |n_2|^{-N_0} |n_3| - 3)^{-N_2} \\
\cdot \|f \chi_{I_{n_1}}\|_{p_1} \|g \chi_{I_{n_2}}\|_{p_2} \|h \chi_{I_{n_3}}\|_{p_3})^r \\lesssim \sum_{|n_1|>10} \sum_{|n_2|>10} \sum_{|n_3|>10} (|n_1| - 6)^{-M_1} |n_2|^{-N_0} |n_3| - 3)^{-N_2} \\
\cdot \|f \tilde{\chi}_{I_0}\|_{p_1} \|g \tilde{\chi}_{I_0}\|_{p_2} \|h \tilde{\chi}_{I_0}\|_{p_3})^r \\lesssim \left( \|f \tilde{\chi}_{I_0}\|_{p_1} \|g \tilde{\chi}_{I_0}\|_{p_2} \|h \tilde{\chi}_{I_0}\|_{p_3} \right)^r.
\]

For other possible choices of $n_1, n_2, n_3$, they will be treated in different ways. Among these cases, when $|n_1| > 10$, we can do similar things as the above to get our desired estimate directly, by considering whether $J$ is close to $I$ or not. Note in the case we are free to take summation over $J$ since we have a decay on $i$ and $j \leq i$.

But when $|n_1| \leq 10$, say $|n_1|, |n_2| \leq 10, |n_3| \geq 10$ things are different. In this situation, the term $\left(1 + \frac{\text{dist}(I_{n_1}, J)}{|J|}\right)^{-M_1}$ in (33) won’t give us a decay factor, which means we will have trouble when taking the summation over dyadic intervals $I$. Actually the decay factors from other terms are with respect to $j$ which can’t help since $i > j$. Recall our desired estimate in this case
\[
\| \sum_{|n_1|,|n_2| \leq 10} \sum_{|n_3|>10} T_1(f \chi_{I_{n_1}}, g \chi_{I_{n_2}}, h \chi_{I_{n_3}})(x) \cdot \varphi_0(x) \|_r \lesssim \|f \tilde{\chi}_{I_0}\|_{p_1} \|g \tilde{\chi}_{I_0}\|_{p_2} \|h \tilde{\chi}_{I_0}\|_{p_3}. \tag{34}
\]
Suppose that from the proof of Theorem 1.2 (see [12, 11]) we can get an additional decay with respect to $n_3$ such like $1/|n_3|^M$ for sufficiently positive integer $M$, then we only need to apply Theorem 1.2 to get
\[
\| \sum_{|n_1|,|n_2| \leq 10} \sum_{|n_3|>10} T_1(f \chi_{I_{n_1}}, g \chi_{I_{n_2}}, h \chi_{I_{n_3}})(x) \cdot \varphi_0(x) \|_r \lesssim \frac{1}{|n_3|^M} \|f \chi_{I_{n_1}}\|_{p_1} \|g \chi_{I_{n_2}}\|_{p_2} \|h \chi_{I_{n_3}}\|_{p_3} \lesssim \|f \tilde{\chi}_{I_0}\|_{p_1} \|g \tilde{\chi}_{I_0}\|_{p_2} \|h \tilde{\chi}_{I_0}\|_{p_3}.
\]
Now we will see how to get such a decay $1/|n_3|^M$. As before we consider two possible cases $\text{dist}(I, J) \leq 3$ and $\text{dist}(I, J) > 3$.

When $\text{dist}(I, J) > 3$, as before consider the integral
\[
\int_{\mathbb{R}} \left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{-M_2} \left(1 + \frac{\text{dist}(x, J)}{|J|}\right)^{-N_3} d\xi.
\]
We can get a decay about $|m|^{-M}$ for $J \subseteq J_m$, $m \in \mathbb{Z}$, and see whether $J_m$ is close to $n_3$ or not. As before by considering whether $J$ is close to $I_{n_3}$ or not, we will get an additional decay $1/|n_3|^M$.

When $\text{dist}(I,J) \leq 3$, as before we have that $J$ is near the origin $J \subseteq 9I_0$. In this case our desired decay comes from the size and energy estimates used in the proof of Theorem 1.2, see [12, 11]. Those size and energy terms corresponding to the function $h\chi_{I_{n_3}}$ would be defined based on the inner product terms like $|\langle \tilde{h} \chi_{I_{n_3}}, \tilde{f}_j \rangle|$. Now since $J$ is close to the origin, such inner product will provide a decay about $1/|n_3|^M$. (Or one can see the proof of Lemma 2.13 or section 8.11 in [12] to see clearly we can actually get such a decay factor for the size estimate.) That means we can get an additional decay from the result of Theorem 1.2, since the boundedness there is based on the size and energy estimates.

So far we have proved Part I in (32).

For Part II, using the intervals $I_n = [n, n + 1]$, $J_m = [m, m + 1]$, $m, n \in \mathbb{Z}$ we can write

$$
\|T_1(f, g, h)(x) \cdot \varphi_0(x)\|_r^r
= \sum_{|n| \geq 5} \sum_{m \in \mathbb{Z}} \sum_{J \subseteq J_m, |J| = 2^{-i}, |J| = 2^{-j}} \frac{1}{|I|^{1/2}} \frac{1}{|J|^{1/2}} \| \langle f, \phi_I \rangle \langle g, \phi_J^1 \rangle \langle h, \phi_J^2 \rangle \phi_J^3(x) \varphi_0(x) \|_r
\lesssim \sum_{|n| \geq 5} \sum_{m \in \mathbb{Z}} \sum_{J \subseteq J_m, |J| = 2^{-i}, |J| = 2^{-j}} \| \frac{1}{|I|^{1/2}} \frac{1}{|J|^{1/2}} \langle f, \phi_I \rangle \langle g, \phi_J^1 \rangle \langle h, \phi_J^2 \rangle \phi_J^3(x) \phi_J^3(x) \varphi_0(x) \|_r.
$$

We will use Hölder’s inequality and take advantage of the decay factors as before to write the above as

$$
\sum_{|n| \geq 5} \sum_{m \in \mathbb{Z}} \sum_{I \subseteq I_n, J \subseteq J_m, |I| = 2^{-i}, |J| = 2^{-j}} \left( \frac{1}{|I|^2} \frac{1}{|J|^2} \right) \left( \| f \tilde{\chi}_{I_n} \|_{p_1} |I|^{p_1-1} \right) \left( \| g \tilde{\chi}_{J_m} \|_{p_2} |J|^{p_2-1} \right) \left( \| h \tilde{\chi}_{J_m} \|_{p_3} |J|^{p_3-1} \right).
$$

(35)
where again $M_j, N_j$ are sufficiently large integers. Then we consider two possible cases, dist$(I_n, J_m) \leq 5$ and dist$(I_n, J_m) > 5$.

When dist$(I_n, J_m) \leq 5$, we use the same technique as before
\[
(|n| - 2)^{-\frac{M_j}{2}} |\tilde{x}_{I_n}| \lesssim |\tilde{x}_{I_0}| \quad \text{and} \quad |\tilde{x}_{I_n}| \sim |\tilde{x}_{J_m}|,
\]
for $M$ sufficiently large. Note that the decay factor for $i$ actually implies a decay for the summation over dyadic intervals $J$, since $i \geq j$. Then we can estimate (35) by
\[
\lesssim \sum_{|n| \geq 5} \left( (|n| - 2)^{-\frac{M_j}{2}} \right) \|f\tilde{x}_{I_n}\|_{p_1} \|g\tilde{x}_{J_m}\|_{p_2} \|h\tilde{x}_{J_m}\|_{p_3})^r
\]
\[
\lesssim \sum_{|n| \geq 5} \left( (|n| - 2)^{-\frac{M_j}{2}} \right) \|f\tilde{x}_0\|_{p_1} \|g\tilde{x}_0\|_{p_2} \|h\tilde{x}_0\|_{p_3})^r
\]
\[
\lesssim \left( \|f\tilde{x}_0\|_{p_1} \|g\tilde{x}_0\|_{p_2} \|h\tilde{x}_0\|_{p_3})^r,\right.
\]
which is the desired estimate.

When dist$(I_n, J_m) > 5$, we need to take advantage of the integral in (35). That is,
\[
\int_{\mathbb{R}} \left( 1 + \frac{\text{dist}(x, I)}{|I|} \right)^{-M_2} \left( 1 + \frac{\text{dist}(x, J)}{|J|} \right)^{-N_3} dx \lesssim |n - m|^{-L},
\]
where $L = \min\{M_2, N_3\}$ is large enough. Now (35) can be written by
\[
\lesssim \sum_{|n| \geq 5} \sum_{|m - n| > 5} \sum_{i, j \geq 0} \sum_{I \subseteq I_n, J \subseteq J_m} \sum_{|I| = 2^{-i}, |J| = 2^{-j}} \left( 2^i (1 + 2^i (|n| - 2))^{-M_j} \right)
\]
\[
\quad \cdot \|f\tilde{x}_{I_n}\|_{p_1} \|g\tilde{x}_{J_m}\|_{p_2} \|h\tilde{x}_{J_m}\|_{p_3} |m - n|^{-L})^r
\]
\[
\lesssim \sum_{|n| \geq 5} \left( (|n| - 2)^{-\frac{M_j}{2}} \right) \|f\tilde{x}_{I_n}\|_{p_1} \|g\tilde{x}_{J_m}\|_{p_2} \|h\tilde{x}_{J_m}\|_{p_3})^r
\]
\[
\lesssim \left( \|f\tilde{x}_0\|_{p_1} \|g\tilde{x}_0\|_{p_2} \|h\tilde{x}_0\|_{p_3})^r,\right.
\]
where as before the decay factor for $i$ allows us to take the summation over dyadic intervals $J$, since $i \geq j$.

Now are are done with Part II, which means we have proved the desired estimate for $T_1(f, g, h)(x)$.

5.3. **Estimates for** $\sum_{k_0 = 100}^{\infty} (2^{-k_0})^j T_{l, k_0}(f, g, h)(x)$.

There is nothing new in this case, since it will be almost the same as what we did for $T_1(f, g, h)(x)$. Note for $T_{l, k_0}(f, g, h)(x)$, the only difference is that we have $|I|^{-1} \sim |\omega_i^2| \sim 2^{k_0} |J|^{-1} \sim |\omega_j^3|$, instead of $|I|^{-1} \sim |\omega_i^2| \geq |J|^{-1} \sim |\omega_j^3|$ in $T_1(f, g, h)(x)$. That is, let $|I| = 2^{-i}, |J| = 2^{-j}$, we will have $i - k_0 = j \geq 0, k_0 \geq 100$. 


Recall we only need $i \geq j$ in the proof for $T_1(f, g, h)(x)$, and the method obviously works for $T_{i,k_0}(f, g, h)(x)$ in the setting $i - k_0 = j \geq 0, k_0 \geq 100$, which will give us a bound uniformly with respect to $k_0$. Then we will be able to take the summation over $k_0$ by using $l \geq 1$. In this way we can get the estimate for \[ \sum_{k_0=100}^{\infty} (2^{-k_0})^l T_{i,k_0}(f, g, h)(x). \]

So far we have proved the desired localized estimate for the operator $T_{ab}^{E,0,0}(f, g, h)(x)$ in (27), which means Theorem 3.1 has been proved. Then from this localized result, we can conclude that Theorem 1.3 is true.

References


