Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness

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**A B S T R A C T**

The main purpose of this paper is three-fold. First of all, we are concerned with the limited smoothness conditions in the spirit of Hörmander on the multi-linear and multi-parameter Coifman–Meyer type Fourier multipliers studied by C. Muscalu, J. Pipher, T. Tao, C. Thiele (2004, 2006) where they established the $L^r$ estimates for the multiplier operators under the assumption that the multiplier has smoothness of sufficiently large order. Under our limited smoothness assumption, we will prove the $L^{p_1} \times \cdots \times L^{p_n} \rightarrow L^r$ boundedness with
\[ \frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r} \]
for $1 < p_1, \ldots, p_n < \infty$ and $0 < r < \infty$. Second, our proof of $L^r$ estimates also offers a different and more direct approach than the one given in Muscalu et al. (2004, 2006) where they use the deep analysis of multi-linear and multi-parameter paraproducts. Third, we also prove a Hörmander type multiplier theorem in the weighted Lebesgue spaces for such operators when the Fourier multiplier is only assumed with limited smoothness.

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1. Introduction

The aim of this paper is to consider the limited smoothness condition on the Fourier multipliers in the multi-parameter and multi-linear setting. This is an analogue of the well-known Hörmander–Mihlin type theorem in the linear and multi-linear cases.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^d)$ denote tempered distributions. The Fourier transform $\hat{f}$ and the inverse Fourier transform $\check{f}$ of $f \in \mathcal{S}(\mathbb{R}^d)$ are defined by
\[
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \quad \text{and} \quad \mathcal{F}^{-1}f(\xi) = \check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi.
\]

In the linear case, we first recall the following Mihlin theorem (see, e.g., [1, Corollary 8.11]):
Theorem 1.1. If a multiplier \( m \in C^{n/2 + 1}(\mathbb{R}^n \setminus \{0\}) \) satisfies the following condition
\[
|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \text{for all } |\alpha| \leq \left[ \frac{n}{2} \right] + 1
\]
then the Fourier multiplier operator \( m(D)f = \mathcal{F}^{-1}[m\hat{f}] \) defined with the symbol \( m(\xi) \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) for all \( 1 < p < \infty \).

On the other hand, Hörmander reformulated and improved Mihlin’s theorem using the Sobolev regularity of the multiplier \[2\]. To describe Hörmander’s theorem, we let \( \psi \in \mathcal{S}(\mathbb{R}^d) \) be a Schwartz function satisfying
\[
\sup \, \text{supp } \psi \subset \left\{ \xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad \sum_{j \in \mathbb{Z}} \psi \left( \frac{\xi}{2^j} \right) = 1, \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}.
\]
For \( s \in \mathbb{R} \), the Sobolev space \( H^s(\mathbb{R}^n) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that
\[
\|f\|_{H^s} \triangleq \|(I - \triangle)^{s/2} f\|_2 < \infty,
\]
where \( (I - \triangle)^{s/2} f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \hat{f}(\xi)] \). Then the Hörmander multiplier theorem says

Theorem 1.2. If \( m \in L^\infty(\mathbb{R}^n) \) satisfies
\[
\sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\psi\|_{H^s(\mathbb{R}^n)} < \infty, \quad \text{for all } s > \frac{n}{2},
\]
where \( \psi \) is the same as in (1.3) when \( d = n \) and \( H^s(\mathbb{R}^n) \) is the Sobolev space, then the Fourier multiplier operator \( m(D) \) defined with the symbol \( m \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) for all \( 1 < p < \infty \).

Clearly, Hörmander’s theorem is stronger than Mihlin’s and the number \( \frac{n}{2} \) cannot be improved in Hörmander’s theorem.

We now turn to the weighted estimates for Fourier multipliers. We first introduce the notion of Muckenhoupt’s \( A_p \) weights \[3\]. Let \( 1 < p < \infty \) and denote \( p' = \frac{p}{p-1} \). We say that a weight \( w \geq 0 \) belongs to the Muckenhoupt class \( A_p(\mathbb{R}^d) \), if
\[
\sup_R \left( \frac{1}{|R|} \int_R w(x)dx \right) \left( \frac{1}{|R|} \int_R w(x)^{1-p'}dx \right)^{p-1} < \infty
\]
where the supremum is taken over all cubes \( R \) in \( \mathbb{R}^n \). We also use the notation \( \|f\|_{L^p_w(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \).

Then, Kurtz and Wheeden \[4\] extended Hörmander’s theorem to weighted Lebesgue spaces and proved the following:

Theorem 1.3. Let \( \frac{n}{2} < s \leq n \) and \( 1 < p < \infty \). Assume \( \frac{n}{2} < p < \infty \) and \( w \in A_p(\mathbb{R}^n) \). If \( m \in L^\infty(\mathbb{R}^n) \) satisfies
\[
\sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\psi\|_{H^s(\mathbb{R}^n)} < \infty,
\]
then the Fourier multiplier operator \( m(D) \) defined with the symbol \( m \) is bounded from \( L^p_w(\mathbb{R}^n) \) to \( L^p_w(\mathbb{R}^n) \) for all \( 1 < p < \infty \).

We now turn to the discussion of multi-linear Coifman–Meyer Fourier multiplier operators. We only state the bilinear case as an example for simplicity of its presentation. For \( m \in L^\infty(\mathbb{R}^{2n}) \), the bilinear Coifman–Meyer Fourier multiplier operator \( T_m \) is defined by
\[
T_m(f, g)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} m(\xi, \eta) e^{i\langle \xi, \eta \rangle} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta
\]
for all \( f, g \in \mathcal{S}(\mathbb{R}^n) \).

Coifman and Meyer \[5–7\] first proved that if \( m \in C^1(\mathbb{R}^{2n} \setminus \{0\}) \) satisfies
\[
|\partial^\alpha \partial^\beta m(\xi, \eta)| \leq C_{\alpha\beta} (|\xi| + |\eta|)^{-|\alpha| - |\beta|}
\]
for all \( |\alpha| + |\beta| \leq L \), where \( L \) is a sufficiently large natural number, then \( T_m \) is bounded from \( L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \) to \( L^r(\mathbb{R}^n) \) for all \( 1 < p, q, r < \infty \) satisfying \( 1/p + 1/q = 1/r \). Results in \[5–7\] have been extended to multi-linear Calderón–Zygmund operators by Kenig and Stein \[8\], Grafakos and Kalton \[9\], Grafakos and Torres \[10, 11\] to include \( 0 < r \leq 1 \) (see also recent work of generalizations to bilinear square functions and vector-valued Calderón–Zygmund operators of Hart \[12\]). However, in many cases where \( m \) has only limited smoothness, we cannot use this result since \( L \) is not an explicit number. Finding the best possible number of \( L \) thus becomes an interesting problem. By reducing the bilinear multiplier operators to linear Calderón–Zygmund operators, Coifman–Meyer obtained the \( L^1 \) estimates under the assumption \( L = 2n + 1 \). In \[10\], the authors also proved the condition (1.7)
with \( L = 2n + 1 \) assures the boundedness of \( T_m \) by using the bilinear \( T_1 \) theorem. However this number seems to be too large in view of the linear case.

Recently, Tomita \([13]\) improved this theorem for multipliers with limited smoothness in terms of the Sobolev regularity. To state the result in \([13]\), for \( m \in L^\infty(\mathbb{R}^{2n}) \), we set \( m_k(\xi, \eta) = m(2^k \xi, 2^k \eta)\psi(\xi_1, \eta_1) \), where \( \psi \) is the same as the \((1.3)\) with \( d = 2n \).

**Theorem 1.4.** Let \( s > n, 1 < p, q, r < \infty \) and \( 1/p + 1/q + 1/r = 1 \). If \( m \in L^\infty(\mathbb{R}^{2n}) \) satisfies
\[
\sup_{k \in \mathbb{Z}} \| m_k \|_{H^s(\mathbb{R}^{2n})} < \infty
\]
then \( T_m \) is bounded from \( L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \) to \( L^r(\mathbb{R}^n) \).

For further improvement in this direction in the case \( 0 < r \leq 1 \) or the case where \( p \) or \( q \) can be smaller than or equal to 1, see the works in Grafakos, Miyachi and Tomita \([14]\), Miyachi and Tomita \([15]\) and Grafakos and Si \([16]\).

Fujita and Tomita \([17]\) considered the weighted norm inequalities for multilinear Fourier multiplier operators, for simplicity we only state their result in the bilinear case.

**Theorem 1.5.** Let \( 1 < p, q < \infty \), \( 1/p + 1/q = 1/r \) and \( n < s \leq 2n \). Assume
\begin{enumerate}[(i)]
\item \( \min\{p, q\} > 2n/s \) and \( w \in A_{\min\{p, q, 2n/s\}} \) or \( 2n/s \) \( \leq \min\{p, q\} \), \( 1 < r < \infty \) and \( w^{1/r} \in A_{\min\{p, q, 2n/s\}} \).
\end{enumerate}

If \( m \in L^\infty(\mathbb{R}^{2n}) \) satisfies
\[
\sup_{k \in \mathbb{Z}} \| m_k \|_{H^s(\mathbb{R}^{2n})} < \infty.
\]
Then \( T_m \) is bounded from \( L^p(w) \times L^q(w) \) to \( L^r(w) \).

This theorem can be understood as bilinear version of the results by Kurtz and Wheeden \([4]\).

Next, we discuss the \( L^r \) estimates for the multi-linear and multi-parameter Fourier multiplier operator in the bilinear and bi-parameter case, Muscalu, Pipher, Tao, and Thiele \([18]\) proved the following

**Theorem 1.6.** Let \( 1 < p, q < \infty \), \( 1/r = 1/p + 1/q \), \( 0 < r < \infty \) and \( m \in L^\infty(\mathbb{R}^{4n}) \) satisfy
\[
|\partial_\xi^{\alpha_1} \partial_\eta^{\beta_1} m(\xi, \eta)| \leq C_{\alpha, \beta} \frac{1}{|\alpha|! |\beta|!} (|\xi_1| + |\eta_1|)^{-|\alpha|} (|\xi_2| + |\eta_2|)^{-|\beta|}
\]
for \( |\alpha|, |\beta| \leq M \), and \( |\alpha_2| + |\beta_2| \leq N \), where \( M, N \) are sufficiently large natural numbers.

Then \( T_m \) is bounded from \( L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n}) \) to \( L^r(\mathbb{R}^{2n}) \), where \( T_m \) is defined by
\[
\begin{aligned}
T_m(f, g)(x_1, x_2) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{4n}} m(\xi, \eta) e^{i\xi_1 x_1 + i\eta_1 y_1 + i\xi_2 x_2 + i\eta_2 y_2} f(\xi_1, \xi_2) g(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2.
\end{aligned}
\]

This theorem was extended to the case of multi-linear and multi-parameter setting in \([19]\). The method of proof of the above theorem in \([18, 19]\) is to decompose the multi-linear and multi-parameter Fourier multiplier operator into discretized multi-linear and multi-parameter paraproducts. By proving the \( L^r \) estimates for the discretized paraproducts, they establish the \( L^r \) estimates for the Fourier multipliers. The difficult part of their proof is in the quasi-Banach case when \( 0 < r \leq 1 \) where the standard duality argument for the paraproducts does not work (see also \([20]\)). Therefore, the authors of \([18, 19]\) established the desired result by using a new duality lemma of \( L^\infty \) for \( 0 < r \leq 1 \), the stopping-time decompositions arguments and multi-linear interpolation. We mention in passing that the endpoint estimates of results in \([18, 19]\) were obtained by Lacey and Metcalfe \([21]\) and \( L^r \) estimates in the above \( \text{Theorem 1.6} \) have also been established recently in the case of multi-linear and multi-parameter pseudo-differential operators by W. Dai and the second author \([22]\). Furthermore, symbolic calculus has been carried out and boundedness of multi-term multi-parameter pseudo-differential operators in the Hörmander classes have been established by Q. Hong and the second author \([23]\). More recently, \( L^p \) estimates for modified bilinear and multi-parameter Hilbert transforms have also been established by W. Dai and the second author in \([24]\), where we address the open question raised in \([18]\).

It is worth noting that the smoothness condition for the Fourier multiplier \( m(\xi_1, \xi_2, \eta_1, \eta_2) \) in \([18, 19]\) requires \( M \) and \( N \) to be sufficiently large. Thus, it is interesting to know what the limited smoothness assumption is on \( m \) to assure the \( L^r \) estimates. This is one of the main purposes of this paper.

To establish the \( L^r \) estimates of the multi-linear and multi-parameter Fourier multipliers with limited smoothness, we need to introduce the two-parameter Sobolev spaces. For \( s_1, s_2 \in \mathbb{R} \), the two-parameter Sobolev space \( H^{s_1, s_2}(\mathbb{R}^{4n}) \) consists of all \( f \in S'(\mathbb{R}^{4n}) \) such that
\[
\| f \|_{H^{s_1, s_2}} = \| (I - \Delta)^{s_1/2, s_2/2} f \|_2 < \infty,
\]
\((1.10)\)
Theorem 1.8. Let m ∈ L∞(Rdn). Set
\[ m_{j,k}(ξ_1, ξ_2, η_1, η_2) = m(2^j ξ_1, 2^k ξ_2, 2^j η_1, 2^k η_2) ψ_1(ξ_1, η_1)ψ_2(ξ_2, η_2). \] (1.11)
where ψ_1, ψ_2 are the same as (1.3) with d = 2n. Let s_1, s_2 > n, s = min{s_1, s_2}, 1 < p, q < ∞, p > 2n/r, q > 2n/s and 1/p + 1/q = 1/r with 0 < r < ∞. If m ∈ L∞(Rdn) satisfies
\[ ∑_{j,k∈Z} ∥m_{j,k}∥_{HF^{s_1/2}L^{q}} < ∞ \] (1.12)
then T_m is bounded from L^p(R^{2n}) × L^q(R^{2n}) to L^r(R^{2n}).

Remark. If we allow the smoothness exponents s_1, s_2 to be close to 2n, then p, q are allowed to be taken in the whole range of 1 < p, q < ∞. Consequently, r can be taken to be all 0 < r < ∞. Therefore, our theorem indeed improves the theorem of Muscalu, Pipher, Tao and Thiele [18] by requiring only limited smoothness and our proof given here provides an alternative one different than that in [18,19].

From the above theorem, we have
Theorem 1.9. Let 1 < p, q < ∞ and 1/p + 1/q = 1/r. If m ∈ C^{2n+1}(R^{2n} \ {0} × R^{2n} \ {0}) satisfies
\[ ∥∂_{ξ_1}^{α_1}∂_{ξ_2}^{α_2}∂_{η_1}^{β_1}∂_{η_2}^{β_2}m(ξ_1, ξ_2, η_1, η_2)∥ \leq C_{α_1,β_1,α_2,β_2} (|ξ_1| + |η_1|)^{−|α_1|−|β_1|} (|ξ_2| + |η_2|)^{−|α_2|−|β_2|} \] (1.13)
for all |α_1| + |β_1| ≤ n + 1, |α_2| + |β_2| ≤ n + 1 and (ξ_1, η_1, ξ_2, η_2) ∈ R^{2n} \ {0} × R^{2n} \ {0}, then T_m is bounded from L^p(R^{2n}) × L^q(R^{2n}) to L^r(R^{2n}).

Finally, we consider the weighted norm inequalities for the bilinear and bi-parameter Fourier multipliers. To this end, we first introduce the notion of product A_p weights (see [25]).

Let 1 < p < ∞. We say that a weight w ≥ 0 belongs to the product Muckenhoupt class A_p(R^n × R^n), if
\[ \sup_{R} \left( \frac{1}{|R|} ∫_R w(x,y)dx dy \right)^{1/p} \left( \frac{1}{|R|} ∫_R w(x,y)^{1−1/p}dx dy \right)^{1−1/p} < ∞ \] (1.14)
where the supremum is taken over all rectangles R = I × J, I and J are both cubes in R^n.

We define A_∞(R^n × R^n) = ∪_{p>1} A_p(R^n × R^n) as usual.

Then we can establish the following
Theorem 1.10. Let 1 < p, q < ∞, 1/p + 1/q = 1/r and n < s_1, s_2 ≤ 2n, s = min{s_1, s_2}. Assume
\[ (i) \ p > 2n/s_1 \quad u_1 ∈ A_{ps_1/2n} \] (1.15)
\[ q > 2n/s_2 \quad u_2 ∈ A_{ps_2/2n} \text{ or} \] (1.16)
\[ (ii) \ \min{p,q} < (2n/s')', \quad 1 < r < ∞ \] (1.17)
\[ u_1^{1−r'} ∈ A_{r's/(2n)}, \quad u_2^{1−r'} ∈ A_{r's/(2n)}. \] (1.18)
If m ∈ L∞(R^{4n}) satisfies
\[ ∑_{j,k∈Z} ∥m_{j,k}∥_{HF^{s_1/2}L^{q}} < ∞, \] (1.19)
then T_m is bounded from L^p(u_1) × L^q(u_2) to L^r(w), where w = u_1^{1/p} u_2^{1/q}.

The statements and their proofs of Theorems 1.7 and 1.9 can be easily generalized to multi-linear and multi-parameter settings. We also remark that the proofs of our main theorems can be viewed as alternative ones different from those given in [18,19]. Moreover, we provide weighted estimates for the multi-linear and multi-parameter Coifman–Meyer multiplier operators considered in [18,19]. We only state these results here and leave the details to the reader.

In general, any collection of n generic vectors ξ_1 = (ξ_j)_{j=1}^n, ..., ξ_n = (ξ_j)_{j=1}^n in R^{2n} generates naturally the following collection of t vectors in R^{4n}:
\[ ξ_1 = (ξ_j)_{j=1}^n, \quad ξ_2 = (ξ_j)_{j=1}^n, \quad ..., \quad ξ_t = (ξ_j)_{j=1}^n. \] (1.20)
Let \( m = m(\xi) = \hat{m}(\xi) \) be a bounded symbol in \( L^\infty(\mathbb{R}^{m_t}) \) that is smooth away from the subspaces \( \{ \xi_1 = 0 \} \cup \cdots \cup \{ \xi_t = 0 \} \) and satisfying

\[
|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_t}^{\alpha_t} m(\xi)| \leq C_{\alpha_1, \ldots, \alpha_t} \prod_{i=1}^{t} |\xi_i|^{-|\alpha_i|} \tag{1.21}
\]

for sufficiently many multi-indices \( \alpha_1, \ldots, \alpha_t \). We will naturally want to investigate the \( L^r \) estimates of the \( n \)-linear multiplier operator \( T_m^{(i)} \) defined by

\[
T_m^{(i)}(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}^{m_t}} m(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i (\xi_1 + \cdots + \xi_n)} d\xi. \tag{1.22}
\]

Thus, we can prove the following \( L^r \) estimates for general \( n \)-linear, \( t \)-parameter multiplier operator \( T_m^{(i)} \) with limited smoothness.

**Theorem 1.10.** Let \( m \in L^\infty(\mathbb{R}^{m_t}) \). Set

\[
m_{j_1, \ldots, j_t}(\hat{\xi}_1, \ldots, \hat{\xi}_t) = (2^j \hat{\xi}_1, \ldots, 2^j \hat{\xi}_t) \Psi(\hat{\xi}_1) \cdots \Psi(\hat{\xi}_t),
\]

where \( \Psi_1, \ldots, \Psi_t \) are the same as in (1.3) with \( d = n \ell \) there. For any \( n \geq 1, t \geq 2 \), the \( n \)-linear, \( t \)-parameter multiplier operator \( T_m^{(i)} \) maps \( L^p(\mathbb{R}^d) \times \cdots \times L^p(\mathbb{R}^d) \) to \( L^r(\mathbb{R}^d) \), provided that \( 1 < p_1, \ldots, p_n < \infty, p_1 > \frac{t \ell}{s_1}, \ldots, p_n > \frac{t \ell}{s_t} \), where \( s_1 > \frac{t \ell}{2} \), \ldots, \( s_t > \frac{t \ell}{2} \) and \( s = \min(s_1, \ldots, s_t) \) and \( \frac{1}{t} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} > 0 \) and the multiplier \( m \) satisfies

\[
\sup_{j_1, \ldots, j_t \in \mathbb{Z}} \|m_{j_1, \ldots, j_t}\|_{L^{p_1, \ldots, p_n}(\mathbb{R}^{m_t})} < \infty.
\]

We can also establish the following weighted estimates.

**Theorem 1.11.** Let \( 1 < p_1, \ldots, p_n < \infty, 1 = \frac{1}{p_1} + \cdots + \frac{1}{p_n} < \frac{t \ell}{s} \) and \( \frac{t \ell}{2} < s_1, \ldots, s_t \leq t \ell, s = \min(s_1, \ldots, s_t) \). Assume one of the following two conditions (i) and (ii) holds, namely,

(i) \( p_j > \frac{t \ell}{s} \), \( w_j \in A_{\frac{p_j}{p}} \), \( j = 1, \ldots, n \), or

(ii) \( \min(p_1, \ldots, p_n) < \left(\frac{t \ell}{s}\right)^r, 1 < r < \infty, w_j^{1-r'} \in A_{\frac{p_j}{p}} \) \( \tag{1.23}\)

\[
If m \in L^\infty(\mathbb{R}^{m_t}) \text{ satisfies }
\sup_{j_1, \ldots, j_t \in \mathbb{Z}} \|m_{j_1, \ldots, j_t}\|_{L^{p_1, \ldots, p_n}(\mathbb{R}^{m_t})} < \infty. \tag{1.25}
\]

Then \( T_m \) is bounded from \( L^{p_1}(w_1) \times \cdots \times L^{p_n}(w_n) \) to \( L^r(w) \), where \( w = w_1^\frac{p}{p_1} \cdots w_n^\frac{p}{p_n} \).

The organization of this paper is as follows: In Section 2 we recall some preliminary facts and give some relevant definitions. In Section 3, we prove Theorem 1.7, namely, the \( L^r \) estimates for the multi-linear and multi-parameter Coifman–Meyer multiplier operators with limited smoothness. In Section 4, we give the proof of Theorem 1.9, i.e., the weighted version of Theorem 1.7.

### 2. Preliminary results

The strong maximal operator \( M_s \) for a function \( f \) on \( \mathbb{R}^{2n} \) is defined by

\[
M_sf(x, y) = \sup_{r_1, r_2 > 0} \frac{1}{r_1^n r_2^n} \int_R |f(u, v)| du dv, \tag{2.1}
\]

where \( R = \{(u, v) \in \mathbb{R}^{2n} | |u - x| < r_1, |v - y| < r_2\} \) and \( f \) is a locally integrable function on \( \mathbb{R}^{2n} \). It is well known that \( M_s \) is bounded on \( L^p(\mathbb{R}^{2n}) \) for all \( 1 < p < \infty \).

**Lemma 2.1.** Let \( \epsilon_1, \epsilon_2 > 0 \). Then there exists a constant \( C > 0 \) such that

\[
\sup_{r_1, r_2 > 0} \left( r_1^n r_2^n \int_{\mathbb{R}^{2n}} \frac{|f(u, v)|}{(1 + r_1|x - u|)^{n+\epsilon_1}(1 + r_2|y - v|)^{n+\epsilon_2}} du dv \right) \leq CM_sf(x, y) \tag{2.2}
\]

for all locally integrable functions \( f \) on \( \mathbb{R}^{2n} \).
Proof. Note that
\[
\int_{1}^{n} \frac{|f(u,v)|}{(1 + r_1|u - v|)^{n+\epsilon_1}(1 + r_2|y - v|)^{n+\epsilon_2}} \, dudv \leq CM_f(x,y)
\]
and
\[
\int_{1}^{n} \frac{|f(u,v)|}{(1 + r_1|u - v|)^{n+\epsilon_1}(1 + r_2|y - v|)^{n+\epsilon_2}} \, dudv \\
\leq \sum_{k=0}^{\infty} \int_{u,v:|u-x|<r_1,|v-y|<r_2} \frac{|f(u,v)|}{(1 + r_1|u - v|)^{n+\epsilon_1}(1 + r_2|y - v|)^{n+\epsilon_2}} \, dudv \\
\leq \sum_{k=0}^{\infty} \frac{1}{(1 + 2k)^{n+\epsilon}} \int_{u,v:|u-x|<2k+1,r_1,|v-y|<2k+1,r_2} |f(u,v)| \, dudv.
\]
Then it follows immediately that
\[
\sup_{r_1,r_2 > 0} \left( \int_{2n} (1 + r_1|u - v|)^{n+\epsilon_1}(1 + r_2|y - v|)^{n+\epsilon_2} \, dudv \right) \leq CM_f(x,y). \quad \square
\]

Using the inequality for vector-valued Hardy–Littlewood maximal functions of C. Fefferman and Stein [26], and the fact that \(M_f(x,y) \leq M_1M_2f(x,y)\), where \(M_1\) and \(M_2\) are the Hardy–Littlewood maximal functions with respect to the \(x\) and \(y\) variables respectively, we have the following inequality for the vector-valued strong maximal functions:

**Lemma 2.2.** Let \(1 < p, q < \infty\). Then there exists a constant \(C > 0\) such that
\[
\left\| \sum_{k \in \mathbb{Z}} (Mf_k)^q \right\|_p^{1/q} \leq C \left\| \sum_{k \in \mathbb{Z}} |f_k|^q \right\|_p
\]
for all sequences \((f_k)_{k \in \mathbb{Z}}\) of locally integrable functions on \(\mathbb{R}^n\).

Using the Littlewood–Paley inequality of \(L^p\) estimates in the product space of R. Fefferman and Stein [27], we can deduce immediately the following

**Lemma 2.3.** Let \(1 < p < \infty\), and let \(\Psi_1, \Psi_2 \in \delta(\mathbb{R}^n)\) be such that \(\text{supp} \Psi_1 \subset \{ \xi \in \mathbb{R}^n : 1/a \leq |\xi| \leq a \}\) for some \(a > 1\), supp \(\Psi_2 \subset \{ \eta \in \mathbb{R}^n : 1/b \leq |\eta| \leq b \}\) for some \(b > 1\). Then there exists a constant \(C > 0\) such that
\[
\left\| \sum_{j,k \in \mathbb{Z}} |\Psi_j(D/2^j)\Psi_k(D/2^k)f|^2 \right\|_p^{1/2} \leq C \|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^{2n}),
\]
where \([\Psi_j(D/2^j)\Psi_k(D/2^k)f](\xi_1, \xi_2) = \mathcal{F}^{-1} \left[ \hat{\Psi}_j(\cdot/2^j)\hat{\Psi}_k(\cdot/2^k) \hat{f}(\cdot, \cdot) \right] (\xi_1, \xi_2)\). Moreover, if \(\sum_{j \in \mathbb{Z}} \Psi_j(\xi/2^j) = 1\) for all \(\xi_j \neq 0\), for \(i = 1, 2\), then
\[
\left\| \sum_{j,k \in \mathbb{Z}} |\Psi_j(D/2^j)\Psi_k(D/2^k)f|^2 \right\|_p^{1/2} \approx \|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^{2n}).
\]

Let \(\phi_0\) be a \(C^\infty\)-function on \([0, \infty)\) satisfying
\[
\phi_0(t) = 1 \quad \text{on } [0, 1/8], \quad \text{supp } \phi_0 \subset [0, 1/4]
\]
we set \(\phi_1(t) = 1 - \phi_0(t)\), and set for \(\xi, \eta \in \mathbb{R}^n\) the following notations:
\[
\phi_{(1)}(\xi, \eta) = \phi_0(|\xi|/|\eta|) \quad \phi_{(2)}(\xi, \eta) = \phi_1(|\eta|/|\xi|)
\]
\[
\phi_{(3)}(\xi, \eta) = (1 - \phi_0(|\xi|/|\eta|))(1 - \phi_1(|\eta|/|\xi|)).
\]

**Lemma 2.4 (17).**

(1) For \((\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\},
\[
\phi_{(1)}(\xi, \eta) + \phi_{(2)}(\xi, \eta) + \phi_{(3)}(\xi, \eta) = 1.
\]
(2) Each $\Phi(I)$ satisfies
\[
|a^{(1)}_{\xi}a^{(2)}_{\eta} \Phi(I)(\xi, \eta)| \leq C_{\alpha_1, \alpha_2}(|\xi| + |\eta|)^{-(|\alpha_1| + |\alpha_2|)}
\] (2.10) for all multi-indices $\alpha_1, \alpha_2$.

(3) $\text{supp } \Phi(3) \subset \{||\xi||/8 \leq |\eta| \leq 8||\xi||\}$, $\text{supp } \Phi(1) \subset \{||\xi|| \leq |\eta|/2\}$ and $\text{supp } \Phi(2) \subset \{|\eta| \leq ||\xi||/2\}$.

With a similar proof to that of Lemma 3.2 in [13] with a little modification, we can obtain the following:

Lemma 2.5. Assume that $m \in C^{N+M}(\mathbb{R}^{2n} \setminus \{0\} \times \mathbb{R}^{2n} \setminus \{0\})$ satisfies
\[
|a^{(1)}_{\xi}a^{(2)}_{\eta}a^{(3)}_{\nu}a^{(4)}_{\sigma} m(\xi_1, \xi_2, \eta_1, \eta_2)| \leq C_{\alpha_1, \alpha_2, \beta_1, \beta_2}(|\xi_1| + |\eta_1|)^{-(|\alpha_1| + |\beta_1|)}(|\xi_2| + |\eta_2|)^{-(|\alpha_2| + |\beta_2|)}
\] (2.11) for all $|\alpha_1| + |\beta_1| \leq N$, $|\alpha_2| + |\beta_2| \leq M$ and $(\xi_1, \eta_1, \xi_2, \eta_2) \in \mathbb{R}^{2n} \setminus \{0\} \times \mathbb{R}^{2n} \setminus \{0\}$, where $N, M$ are non-negative integers. Let $\Phi_1$ and $\Phi_2 \in \mathcal{S}(\mathbb{R}^{2n})$ be such that none of supp $\Phi_1$, supp $\Phi_2$ contains the origin, and set
\[
\tilde{m}_{\xi, \eta}(\xi_1, \xi_2, \eta_1, \eta_2) = m(s\xi_1, t\xi_2, s\eta_1, t\eta_2)\Phi_1(\xi_1, \eta_1)\Phi_2(\xi_2, \eta_2).
\] (2.12)

Then $\sup_{s,t>0} \|\tilde{m}_{\xi, \eta}\|_{H^{N,M}(\mathbb{R}^{2n})} < \infty$.

Lemma 2.6 ([14]). Let $2 \leq q < \infty$, $r > 0$ and $s > 0$. Then there exists a constant $C > 0$ such that
\[
\|\tilde{f}\|_{L^q(U_{r,s})} \leq C \|f\|_{L^q(U_{r,s})}^{1/q}
\]
\[
\leq C \|f\|_{H^{s,t}(\mathbb{R}^{2n} \times \mathbb{R}^{2n})},
\] (2.13)

Next, we need to establish the following

Lemma 2.7. Let $s_1, s_2 \in \mathbb{R}$ and let $\Psi_1, \Psi_2 \in \mathcal{S}(\mathbb{R}^{2n})$ be such that supp $\Psi_1$, supp $\Psi_2$ are compact and none of them contains the origin. Assume that $\Psi \in C^\infty(\mathbb{R}^{2n} \setminus \{0\} \times \mathbb{R}^{2n} \setminus \{0\})$ satisfies
\[
|a^{(1)}_{\xi}a^{(2)}_{\eta}a^{(3)}_{\nu} a^{(4)}_{\sigma} \Psi(\xi_1, \xi_2, \eta_1, \eta_2)| \leq C_{\alpha_1, \alpha_2, \beta_1, \beta_2}(|\xi_1| + |\eta_1|)^{-(|\alpha_1| + |\beta_1|)}(|\xi_2| + |\eta_2|)^{-(|\alpha_2| + |\beta_2|)}
\] (2.11) for all $|\alpha_1| + |\beta_1| \leq N$, $|\alpha_2| + |\beta_2| \leq M$ and $(\xi_1, \eta_1, \xi_2, \eta_2) \in \mathbb{R}^{2n} \setminus \{0\} \times \mathbb{R}^{2n} \setminus \{0\}$, where $N, M$ are non-negative integers. Let $\Phi_1$ and $\Phi_2 \in \mathcal{S}(\mathbb{R}^{2n})$ be such that none of supp $\Phi_1$, supp $\Phi_2$ contains the origin, and set
\[
\tilde{m}_{s_1, s_2}(\xi_1, \xi_2, \eta_1, \eta_2) = m(s\xi_1, t\xi_2, s\eta_1, t\eta_2)\Psi_1(\xi_1, \eta_1)\Psi_2(\xi_2, \eta_2).
\] (2.12)

Then $\sup_{s,t>0} \|\tilde{m}_{s_1, s_2}\|_{H^{s_1, s_2}(\mathbb{R}^{2n})} < \infty$.

By Lemma 2.5, $\sup_{s,t>0} \|\Phi(\xi, \eta)\|_{H^{s_1, s_2}} < \infty$.

The proof is then complete. \qed
3. Proof of Theorem 1.7

The main effort of this section is to establish the first main theorem of this paper on $L'$ estimates for the multi-linear and multi-parameter Fourier multipliers with limited smoothness, namely, Theorem 1.7. The proof is quite complicated and involved due to the multi-parameter structure of the Fourier multiplier $m$. Therefore, we will divide the proof into several steps. The main idea is to decompose the multiplier into different pieces and handle them separately in each piece.

**Proof.** Let $s_1, s_2 > n$ and $m \in L^\infty(\mathbb{R}^{dn})$ satisfy $\sup_{j, k \in \mathbb{Z}} \| m_{j, k} \|_{H^{s_1, s_2}} < \infty$, where $m_{j, k}$ is defined by (1.11). Since $H^{s_1, s_2}(\mathbb{R}^{dn}) \hookrightarrow H^{\min\{s_1, s_2\}, \min\{s_1, s_2\}}(\mathbb{R}^{dn})$, it is sufficient to consider $H^{s_1, s_2}(\mathbb{R}^{dn})$, where $s = \min\{s_1, s_2\} > n$. We rewrite $m$ as follows:

$$m(\xi_1, \xi_2, \eta_1, \eta_2) = m(\xi_1, \xi_2, \eta_1, \eta_2) \left( \sum_{i=1}^{3} \Phi_i(\xi_1, \eta_1) \right) \left( \sum_{j=1}^{3} \Phi_j(\xi_2, \eta_2) \right)$$

$$= \sum_{i, j=1}^{3} m(\xi_1, \xi_2, \eta_1, \eta_2) \Phi_i(\xi_1, \eta_1) \Phi_j(\xi_2, \eta_2)$$

$$= \sum_{i, j=1}^{3} m_{i,j}(\xi_1, \xi_2, \eta_1, \eta_2) \tag{3.1}$$

where $\Phi_i, \Phi_j$ $(1 \leq i, j \leq 3)$ are defined by (2.7) and (2.8).

By Lemma 2.4, we divide these $m_{j, k}$ into four groups and estimate the bilinear and bi-parameter Fourier multiplier operator defined by each symbol $m_{j, k}$. Since the Fourier multiplier operator corresponding to every symbol $m_{j, k}$ in the same group can be estimated in the similar way, we just choose one to handle in each group.

- **Group 1:**
  - $m_{1,1}$, where $\text{supp } m_{1,1} \in \{ |\xi_1| \leq |\eta_1|/2, |\xi_2| \leq |\eta_2|/2 \}$
  - $m_{2,2}$, where $\text{supp } m_{2,2} \in \{ |\xi_1| \leq |\xi_2|/2, |\eta_1| \leq |\xi_2|/2 \}$.

- **Group 2:**
  - $m_{1,2}$, where $\text{supp } m_{1,2} \in \{ |\xi_1| \leq |\eta_1|/2, |\eta_2| \leq \frac{1}{2} |\xi_2| \}$
  - $m_{2,3}$, where $\text{supp } m_{2,3} \in \{ |\eta_1| \leq |\xi_1|/2, |\eta_2| \leq \frac{1}{2} |\xi_2| \}$
  - $m_{3,1}$, where $\text{supp } m_{3,1} \in \{ |\xi_1|/2 \leq |\xi_1| \leq 8 |\eta_1|, |\xi_2| \leq |\eta_2|/2 \}$
  - $m_{3,2}$, where $\text{supp } m_{3,2} \in \{ |\eta_1|/2 \leq |\xi_1| \leq 8 |\eta_1|, |\eta_2| \leq |\eta_2|/2 \}$.

- **Group 3:**
  - $m_{1,2}$, where $\text{supp } m_{1,2} \in \{ |\eta_1| \leq |\eta_1|/2, |\eta_2| \leq |\xi_2|/2 \}$
  - $m_{2,1}$, where $\text{supp } m_{2,1} \in \{ |\eta_1| \leq |\xi_1|/2, |\xi_2| \leq |\xi_2|/2 \}$.

- **Group 4:**
  - $m_{3,3}$, where $\text{supp } m_{3,3} \in \{ |\eta_1|/2 \leq |\xi_1| \leq 8 |\eta_1|, |\eta_2|/2 \leq |\xi_2| \leq 8 |\eta_2| \}$.

In the following proof, we assume $2n/s < p, q$.

**Estimates for Fourier multiplier corresponding to a symbol $m_{j, k}$ in Group 1.**

First, we consider $m_{2,2}$, for simplicity we denote it as $m^1$ instead of $m_{2,2}$. Using the fact that $L^p$ norm is bounded by the $H^p$ norm in the multi-parameter setting established, e.g., in [28–30], and the equivalence of the definition of the multi-parameter Hardy space, we have for all $0 < r < \infty$

$$\| T_m(f, g) \|_{L^p} \leq \| \sup_{k, t > 0} |\hat{\Phi}_{k, t} \ast T_m(f, g)| \|_{L^p}$$

$$\approx \left\| \left\{ \sum_{j, k \in \mathbb{Z}} \left| \psi_j(D/2^j) \psi_k(D/2^k) T_m(f, g) \right|^2 \right\}^{1/2} \right\|_{L^p}$$

(3.2)

for $0 < p < \infty$, where $\Phi_{k, t}(x, y) = 2^{sn} \hat{\phi}(2^{sn} x) 2^{tn} \hat{\phi}(2^{tn} y), \phi \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\phi}$ does not contain the origin, $\psi$ is the same as (1.3) with $d = n$.

Let $f, g \in \mathcal{S}(\mathbb{R}^{2n})$, since $\sum_{j \in \mathbb{Z}} \psi_j(\xi) = 1$, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$A_{j, k} \triangleq \psi(D/2^j) \psi(D/2^k) T_{m^1}(f, g)(x_1, x_2)$$

$$= \frac{1}{(2\pi)^{(dn)}} \int_{\mathbb{R}^{dn}} m^1(\xi_1, \xi_2, \eta_1, \eta_2) e^{i\xi_1(\xi_1 + \eta_1) + i\xi_2(\xi_2 + \eta_2)} \psi_j(\xi_1 + \eta_1) \hat{f}(\xi_1, \xi_2) \psi_k(\xi_2 + \eta_2) \hat{g}(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

$$= \frac{1}{(2\pi)^{(dn)}} \int_{\mathbb{R}^{dn}} m^1(\xi_1, \xi_2, \eta_1, \eta_2) e^{i\xi_1(\xi_1 + \eta_1) + i\xi_2(\xi_2 + \eta_2)}$$

$$\times \psi_j(\xi_1 + \eta_1) \hat{\psi}_j(\xi_1) \hat{f}(\xi_1, \xi_2) \psi_k(\xi_2 + \eta_2) \hat{\psi}_k(\xi_2) \hat{g}(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$
\[
\begin{align*}
&= \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} m^1(\xi_1, \xi_2, \eta_1, \eta_2) e^{i\mathbf{k} \cdot (\xi_1 + \eta_1 + \mathbf{k} \cdot (\xi_2 + \eta_2))} \\
&\quad \times \tilde{\Psi}(\xi_1 + \eta_1) \tilde{\Psi}(\xi_2) \Psi_0(\xi_2 + \eta_2) \tilde{\Psi}_k(\xi_2) \tilde{\Psi}(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2 \\
&= \int_{\mathbb{R}^{4n}} 2^{(2n+2k)n} (F^{-1}m^1_{j,k})(2'(x_1 - y_1), 2'(x_2 - y_2), 2'(x_1 - z_1), 2'(x_2 - z_2)) \\
&\quad \times (\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f(y_1, y_2)g(z_1, z_2) dy_1 dy_2 dz_1 dz_2)
\end{align*}
\]

where \( \Psi_k(\xi) = \Psi(\xi / 2^k) \) and \( \tilde{\Psi}(\xi) \in \mathcal{S}(\mathbb{R}^n) \) such that \( \tilde{\Psi}(\xi_1)\Psi(\xi_1 + \eta_1) = \Psi(\xi_1 + \eta_1) \), on the supp \( m^1 \), since \( |\xi_1 + \eta_1| \approx |\xi_1| \).

The same is true for \( \tilde{\Psi}(\xi_2) \), i.e., \( \Psi_k(\xi_2 + \eta_2) = \Psi(\xi_2 + \eta_2) \), on the supp \( m^1 \), since \( |\xi_2 + \eta_2| \approx |\xi_2| \).

\[ m^1_{j,k} = m^1(2^j\xi_1, 2^j\xi_2, 2^j\eta_1, 2^j\eta_2) \Psi(\xi_1 + \eta_1) \Psi(\xi_2 + \eta_2). \]

Take \( 1 < t < 2 \) satisfying \( 2n/s < t < \min\{2, p, q\} \).

\[ |A_{j,k}| \leq 2^{2(n+2k)n} \int_{\mathbb{R}^{4n}} \left( 1 + 2^j|x_1 - y_1| + 2^j|x_1 - z_1| \right)^s \left( 1 + 2^j|x_2 - y_2| + 2^j|x_2 - z_2| \right)^s \]

\[ \times \left( (F^{-1}m^1_{j,k})(2'(x_1 - y_1), 2'(x_2 - y_2), 2'(x_1 - z_1), 2'(x_2 - z_2)) \right)^s \]

\[ \times \left( \int_{\mathbb{R}^{4n}} (1 + |y_1| + |z_1|) \left| (1 + 2^j|x_2 - y_2| + 2^j|x_2 - z_2|)^s \right| \left| (1 + 2^j|x_2 - y_2| + 2^j|x_2 - z_2|)^s \right| \right)^{1/t} \]

\[ \leq \left\| m^1_{j,k} \right\|_{1^{s/t}} \left( \int_{\mathbb{R}^{2n}} 2^{2(n+2k)n} (g(z_1, z_2))^s (1 + 2^j|x_2 - z_2|)^{-st/2} (1 + 2^j|x_1 - z_1|)^{-st/2} dz_1 dz_2 \right)^{1/t} \]

\[ \times \left( \int_{\mathbb{R}^{2n}} 2^{2(n+2k)n} (\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f(y_1, y_2))^s (1 + |y_1|)^{-st/2} (1 + 2^j|x_2 - y_2|)^{-st/2} dy_1 dy_2 \right)^{1/t} \]

\[ \leq \left\| m^1_{j,k} \right\|_{1^{s/t}} \left( M_s((\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)^s)(x_1, x_2) \right)^{1/t} \left( M_s((g)^s)(x_1, x_2) \right)^{1/t}. \]

The last inequality is from Lemmas 2.1 and 2.7 since \( st/2 > n \).

Then by Hölder’s inequality, (3.2) and (3.5), we have

\[ \left\| T_{m^1}(f, g)(x_1, x_2) \right\|_l' \leq \sup_{j,k \in \mathbb{Z}} \left\| m^1_{j,k} \right\|_{H^{s/t}} \left[ \sum_{j,k} (M_s((\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)^s)) \right]^{1/2} \left\| (M_s((g)^s)) \right\|^{1/2}_{l^q} \]

\[ \leq \sup_{j,k \in \mathbb{Z}} \left\| m^1_{j,k} \right\|_{H^{s/t}} \left[ \sum_{j,k} (M_s((\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)^s)) \right]^{1/2} \left\| (M_s((g)^s)) \right\|^{1/2}_{l^q} \]

\[ \leq \sup_{j,k \in \mathbb{Z}} \left\| m^1_{j,k} \right\|_{H^{s/t}} \left\| f \right\|_l \left\| g \right\|_{l^q}. \]

Using \( \sup m^1 \in \{1/a \leq \sqrt{|\xi_1|^2 + |\eta_1|^2} \leq a, 1/b \leq \sqrt{|\xi_2|^2 + |\eta_2|^2} \leq b \} \) for some \( a, b > 1 \), by Lemma 2.7 we have

\[ \sup_{j,k \in \mathbb{Z}} \left\| m^1_{j,k} \right\|_{H^{s/t}} \leq \sup_{j,k \in \mathbb{Z}} \left\| m^i_{j,k} \right\|_{H^{s/t}} \]

\[ \sup_{j,k \in \mathbb{Z}} \left\| m^i_{j,k} \right\|_{H^{s/t}} \]

Consequently

\[ \left\| T_{m^1} \right\|_{l^p \times l^q \rightarrow l'} \leq \sup_{j,k \in \mathbb{Z}} \left\| m^i_{j,k} \right\|_{1^{s/t}}. \]

Changing the roles \( \xi_1, \eta_1 \) and \( \xi_2, \eta_2 \), we can prove

\[ \left\| T_{m^1} \right\|_{l^p \times l^q \rightarrow l'} \leq \sup_{j,k \in \mathbb{Z}} \left\| m^i_{j,k} \right\|_{H^{s/t}} \]

where \( m^1 = m_{1,1} \) this time.
Estimates for the Fourier multiplier operators with a symbol in Group 2:

We write $m^2$ instead of $m_{1,3}$ for simplicity. Since $\text{supp } m_{1,3} \subset \{(|\xi_1| \leq |\eta_1|/2, |\eta_2|/8 \leq |\xi_2| \leq 8|\eta_2|\}$, then there exists $\Psi \in \mathcal{S}(\mathbb{R}^n)$, such that $\Psi(\xi_2)\Psi^2(\xi_2) = \Psi(\xi_2)$ on $\{(|\eta_2|/8 \leq |\xi_2| \leq 8|\eta_2|\}$, where $\Psi$ is the function which is the same as case 1. Hence,

\[
\psi(D/2^j) T_{m^2}(f, g)(x_1, x_2) = \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} m^2(\xi_1, \xi_2, \eta_1, \eta_2) e^{i\xi_1(\xi_1 + \eta_1)} + i\xi_2(\xi_1 + \eta_2) \psi_j(\xi_1 + \eta_1) \hat{f}(\xi_2) \hat{g}(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]

\[
= \frac{1}{(2\pi)^{4n}} \sum_k \int_{\mathbb{R}^{4n}} m^2(\xi_1, \xi_2, \eta_1, \eta_2) e^{i\xi_1(\xi_1 + \eta_1)} e^{i\xi_2(\xi_2 + \eta_2)}
\times \psi_j(\xi_1 + \eta_1) \hat{\psi}_j(\eta_1) \psi_j(\xi_2) \hat{f}(\xi_1, \xi_2) \psi_j(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]

\[
= \frac{1}{(2\pi)^{4n}} \sum_k \int_{\mathbb{R}^{4n}} m^2(\xi_1, \xi_2, \eta_1, \eta_2) e^{i\xi_1(\xi_1 + \eta_1)} e^{i\xi_2(\xi_2 + \eta_2)}
\times \psi_j(\xi_1 + \eta_1) \psi_k(\xi_2) \psi_k(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]

\[
= \sum_k \int_{\mathbb{R}^{4n}} 2(2\pi)^n(2^j m^2_{\xi, \eta} f(x_1 - y_1), 2^k (x_2 - y_2), 2^l (x_1 - z_1), 2^k (x_2 - z_2))
\times (\hat{\psi}_j(D) \hat{\psi}_k(D) f)(y_1, y_2) (\psi_k(D) g)(z_1, z_2) dy_1 dy_2 dz_1 dz_2
\]

\[
\Psi(D/2^j) T_{m^2}(f, g)(x_1, x_2) = \frac{1}{(2\pi)^{4n}} \sum_k \int_{\mathbb{R}^{4n}} m^2(\xi_1, \xi_2, \eta_1, \eta_2) e^{i\xi_1(\xi_1 + \eta_1)} e^{i\xi_2(\xi_2 + \eta_2)}
\times \psi_j(\xi_1 + \eta_1) \psi_k(\xi_2) \psi_k(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]

\[
\leq \sum_k A_{j,k}
\]  

(3.10)

where $\hat{\psi}$ is the same as we used in Estimates for symbols in Group 1 and $\Psi(\xi_2)\Psi^2(\xi_2) = \Psi(\xi_2)$.

\[
m^2_{\xi, \eta} = m^2(2^j \xi_1, 2^k \eta_1, 2^l \xi_2, 2^k \eta_2) \Psi(\xi_1 + \eta_1) \Psi(\xi_2).
\]  

(3.11)

Take $1 < t < 2$ satisfying $2n/s < t < \min\{2, p, q\}$. Arguing in the same way as deriving (3.5), we can prove

\[
|A_{j,k}| \lesssim \|m^2_{\xi, \eta}\|_{H^{s,t}} \left( M(|(\hat{\psi}_j(D) \hat{\psi}_k(D) f)|^t)(x_1, x_2) \right)^{1/t} \left( M(|\Psi^2(D) g|^t)(x_1, x_2) \right)^{1/t}.
\]

(3.12)

Moreover we can assume $f(\xi_1, \xi_2) = f_1(\xi_1) f_2(\xi_2)$, where $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, since $f_1 \otimes f_2$ is dense in $L^p(\mathbb{R}^{2n})$, $1 < p < \infty$. Then we have

\[
|A_{j,k}| \lesssim \|m^2_{\xi, \eta}\|_{H^{s,t}} \left( M(|g_1|^t)(x_1) M(|(\hat{\psi}_j(D) f_1)|^t)(x_1) \right)^{1/t} \left( M(|\Psi^2(D) g_2|^t)(x_2) M(|\Psi^2(D) f_2|^t)(x_2) \right)^{1/t}.
\]

(3.13)

Then from (3.10) and (3.13), we have

\[
\|\psi(D/2^j) T_{m^2}(f, g)(x_1, x_2)\| \lesssim \sum_k \|m^2_{\xi, \eta}\|_{H^{s,t}} \left( M(|g_1|^t)(x_1) M(|(\hat{\psi}_j(D) f_1)|^t)(x_1) \right)^{1/t}
\times \left( M(|\Psi^2(D) g_2|^t)(x_2) M(|\Psi^2(D) f_2|^t)(x_2) \right)^{1/t}
\]

\[
\times \left\{ \sum_k \left[ M(|\Psi^2(D) g_2|^t)(x_2) M(|\Psi^2(D) f_2|^t)(x_2) \right]^{1/t} \right\}.
\]

(3.14)

Then

\[
\left( \sum |\psi(D/2^j) T_{m^2}(f, g)(x_1, x_2)|^2 \right)^{1/2} \lesssim \sup_{j, k \in \mathbb{Z}} \|m^2_{\xi, \eta}\|_{H^{s,t}} \left\{ \sum_j \left[ M(|g_1|^t)(x_1) M(|(\hat{\psi}_j(D) f_1)|^t)(x_1) \right]^{2/\ell}
\times \left[ \sum_k \left( M(|\Psi^2(D) g_2|^t)(x_2) M(|\Psi^2(D) f_2|^t)(x_2) \right)^{1/\ell} \right]^{2/\ell} \right\}^{1/2}
\]

\[
= \sup_{j, k \in \mathbb{Z}} \|m^2_{\xi, \eta}\|_{H^{s,t}} \left\{ \sum_j \left[ M(|g_1|^t)(x_1) M(|(\hat{\psi}_j(D) f_1)|^t)(x_1) \right]^{2/\ell}
\times \left[ \sum_k \left( M(|\Psi^2(D) g_2|^t)(x_2) M(|\Psi^2(D) f_2|^t)(x_2) \right)^{2/\ell} \right]^{1/\ell} \right\}^{1/2}.
\]

(3.15)
Since \(p/t, q/t, 2/t > 1\), by Hölder’s inequality, Lemmas 2.2, 2.3 and (3.15)

\[
\left\| T_m^2(f, g)(x_1, x_2) \right\|_{L^t} \lesssim \left\| \left( \sum_j \left| \Psi(D/2^j)^2 T_m^2(f, g)(x_1, x_2) \right|^2 \right)^{1/2} \right\|_{L^t(\mathbb{R}^n)}
\]

\[
\lesssim \sup_{j, k \in \mathbb{Z}} \| m_j^2 \|_{H^{s+t}} \left\| \left( \sum_k \left[ \left| M(|\psi_k|^2)(x_2) \right| \right]^2 \right)^{1/2} \right\|_{L^t(\mathbb{R}^n)}
\]

\[
\times \left\| \left( \sum_k \left[ \left| M(|\psi_k|^2)(x_2) \right| \right]^2 \right)^{1/2} \right\|_{L^t(\mathbb{R}^n)}
\]

\[
\lesssim \sup_{j, k \in \mathbb{Z}} \| m_j^2 \|_{H^{s+t}} \left\| \left( \sum_k \left[ \left| M(|\psi_k|^2)(x_2) \right| \right]^2 \right)^{1/2} \right\|_{L^t(\mathbb{R}^n)}
\]

\[
\times \left\| \left( \sum_k \left[ \left| M(|\psi_k|^2)(x_2) \right| \right]^2 \right)^{1/2} \right\|_{L^t(\mathbb{R}^n)}
\]

\[
\lesssim \sup_{j, k \in \mathbb{Z}} \| m_j^2 \|_{H^{s+t}} \left\| f_1 \right\|_{L^p} \| g_1 \|_{L^q}
\]

\[
\times \left\| \left( \sum_k \left[ \left| M(|\psi_k|^2)(x_2) \right| \right]^2 \right)^{1/2} \right\|_{L^t(\mathbb{R}^n)}
\]

\[
\lesssim \sup_{j, k \in \mathbb{Z}} \| m_j^2 \|_{H^{s+t}} \left\| f_1 \right\|_{L^p} \| f_2 \|_{L^p} \| g_1 \|_{L^q} \| g_2 \|_{L^q}
\]

\[
\text{(3.16)}
\]

Using \(\sup m_j^2 \in \{1/a \leq 2\xi_1^2 + |\eta_1|^2 \leq a, 1/b \leq 2\xi_2^2 + |\eta_2|^2 \leq b\}\) for some \(a, b > 1\), by Lemma 2.7 we have

\[
\sup_{j, k \in \mathbb{Z}} \| m_j^2 \|_{H^{s+t}} \lesssim \sup_{j, k \in \mathbb{Z}} \| m_{j, k} \|_{H^{s+t}}.
\]

\[
\text{(3.17)}
\]

Consequently

\[
\| T_m^2 \|_{L^p(\mathbb{R}^n) \rightarrow L^t} \leq \sup_{j, k \in \mathbb{Z}} \| m_{j, k} \|_{H^{s+t}}.
\]

\[
\text{(3.18)}
\]

By changing the roles of \(x_1\) and \(x_2\) or \((x_1, \eta_1)\) and \((x_2, \eta_2)\), we can prove other situations in Group 2. Estimates for Fourier multiplier with symbols in Group 3:

We write \(m^3\) instead of \(m_{1, 2}\), the proof is similar to case 1 with necessary modification. Since \(\xi_1 + \eta_1 \approx |\eta_1|\) and \(\xi_2 + \eta_2 \approx |\xi_2|\), we have

\[
\Psi(D/2^j)^2 \Psi(D/2^k) T_{m^3}(f, g)(x_1, x_2) = \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} \left| m^3(\xi_1, \xi_2, \eta_1, \eta_2) \right| e^{ix_1(\xi_1 + \eta_1) + ix_2(\xi_2 + \eta_2)}
\]

\[
\times \bar{\Psi}_i(\xi_1 + \eta_1) \bar{\Psi}_k(\xi_2 + \eta_2) \bar{\Psi}_j(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]

\[
= \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} \left| m^3(\xi_1, \xi_2, \eta_1, \eta_2) \right| e^{ix_1(\xi_1 + \eta_1) + ix_2(\xi_2 + \eta_2)}
\]

\[
\times \bar{\Psi}_i(\xi_1 + \eta_1) \bar{\Psi}_k(\xi_2 + \eta_2) \bar{\Psi}_j(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]

\[
= \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} \left| m^3(\xi_1, \xi_2, \eta_1, \eta_2) \right| e^{ix_1(\xi_1 + \eta_1) + ix_2(\xi_2 + \eta_2)}
\]

\[
\times \bar{\Psi}_i(\xi_1 + \eta_1) \bar{\Psi}_k(\xi_2 + \eta_2) \bar{\Psi}_j(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]

\[
= \int_{\mathbb{R}^{4n}} 2^{2n+2kn} (2^{-1} m^3_{j, k})(2^j(x_1 - y_1), 2^k(x_2 - y_2), 2^j(x_1 - z_1), 2^k(x_2 - z_2))
\]

\[
\times (\bar{\Psi}_i(D) f)(y_1, y_2) \bar{\Psi}_j(D) g(z_1, z_2) dy_1 dy_2 dz_1 dz_2
\]

\[
\triangleq A_{j, k}
\]

\[
\text{(3.19)}
\]

where \(\Psi, \tilde{\Psi}\) are defined the same way as we deal with symbols in Group 1 and

\[
m^3_{j, k} = m^3(2^j \xi_1, 2^k \xi_2, 2^j \eta_1, 2^k \eta_2) \Psi(\xi_1 + \eta_1) \Psi(\xi_2 + \eta_2).
\]

\[
\text{(3.20)}
\]
As we did in dealing with symbols in Group 1, we can easily prove

$$|A_{j,k}| \lesssim \|m_{j,k}^3\|_{H^{s,t}}(M_s(|\Psi_j(D)f|)^4)(x_1, x_2))^{1/(4s)}(M_s(|\Psi_k(D)g|)^4)(x_1, x_2))^{1/t}$$

(3.21)

where max(1, 2n/s) < t < 2.

Since the rest of the proof is similar to that of case 1, we omit the details. Thus we obtain

$$\|T_{m^4}\|_{L^p \rightarrow L^r} \lesssim \sup_{j,k \in Z} \|m_{j,k}^3\|_{H^{s,t}} \leq \sup_{j,k \in Z} \|m_{j,k}\|_{H^{s,t}}.$$  

(3.22)

By changing the roles of ($\xi_1, \eta_1$) and ($\xi_2, \eta_2$), we can get the same conclusion for $m_{2,1}$.

Estimates for Fourier multipliers with symbols in Group 4:

We write $m^4$ instead of $m_{2.3}$. Since the proof is similar to the case dealing with symbols in Group 2, we will outline the main estimates and omit the details here.

First, we can easily prove

$$|T_{m^4}(f, g)(x_1, x_2)| \lesssim \sup_{j,k \in Z} \|m_{j,k}^4\|_{H^{s,t}} \left\{ \sum_{j,k} (M_s(|\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f|)^4)(x_1, x_2) \right\}^{1/2} \times \left\{ \sum_{j,k} (M_s(|\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)g|)^4)(x_1, x_2) \right\}^{1/2}$$

(3.23)

where max(1, 2n/s) < t < 2.

$$m_{j,k}^4 = m^4(2^j \xi_1, 2^j \eta_1, 2^k \xi_2, 2^k \eta_2) \Psi_j(\xi_1 + \eta_1)\Psi_k(\xi_2 + \eta_2)\Psi_j(\xi_2).$$

(3.24)

Since $p/t, q/t, 2/t > 1$, by Hölder's inequality, Lemmas 2.2 and 2.3, we have

$$\|T_{m^4}(f, g)(x_1, x_2)\|_{L^r} \lesssim \sup_{j,k \in Z} \|m_{j,k}^4\|_{H^{s,t}} \left\{ \sum_{j,k} (M_s(|\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f|)^{2/1}) \right\}^{1/2} \times \left\{ \sum_{j,k} (M_s(|\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)g|)^{2/1}) \right\}^{1/2}$$

$$\lesssim \sup_{j,k \in Z} \|m_{j,k}^4\|_{H^{s,t}} \left\{ \sum_{j,k} \left( \frac{\|\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f\|_{L^p} \|\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)g\|_{L^q}}{\|f\|_{L^p} \|g\|_{L^q}} \right)^{2/1} \right\}^{1/2}$$

(3.25)

Since supp $m^4 \in \{1/a \leq \sqrt{|\xi_1|^2 + |\eta_1|^2} \leq a, 1/b \leq \sqrt{|\xi_2|^2 + |\eta_2|^2} \leq b\}$ for some $a, b > 1$, by Lemma 2.7 we have

$$\|T_{m^4}\|_{L^p \rightarrow L^r} \lesssim \sup_{j,k \in Z} \|m_{j,k}^4\|_{H^{s,t}} \lesssim \sup_{j,k \in Z} \|m_{j,k}\|_{H^{s,t}}.$$  

(3.26)

Next, we consider $T_{m^1}, T_{m^2}$, the dual operator of $T_m$, which are defined by

$$\int_{\mathbb{R}^{2n}} T_m(f, g)hdx = \int_{\mathbb{R}^{2n}} T_{m^1}(h, g)f dx = \int_{\mathbb{R}^{2n}} T_{m^2}(f, h)g dx$$

(3.27)

for all $f, g, h \in \mathcal{S}(\mathbb{R}^{2n})$.

If we have proved the same conclusion for $T_{m^1}, T_{m^2}$ as $T_m$, then using the same proof as in the bilinear case in [13], we complete the proof of Theorem 1.7 by multi-linear and multi-parameter duality and interpolation. We omit the details here.
To finish the proof of Theorem 1.7, we only need to show
\[
\begin{align*}
\sup_{j,k \in \mathbb{Z}} \|m_{j,k}^1\|_{H^{1/2}((\mathbb{R}^n)^* \backslash \mathbb{R}^n)} & \leq \sup_{j,k \in \mathbb{Z}} \|m_{j,k}^2\|_{H^{1/2}((\mathbb{R}^n)^* \backslash \mathbb{R}^n)} \\
\sup_{j,k \in \mathbb{Z}} \|m_{j,k}^3\|_{H^{1/2}((\mathbb{R}^n)^* \backslash \mathbb{R}^n)} & \leq \sup_{j,k \in \mathbb{Z}} \|m_{j,k}^4\|_{H^{1/2}((\mathbb{R}^n)^* \backslash \mathbb{R}^n)}
\end{align*}
\]
(3.28)
where \(m^1(\xi_1, \eta_1, \xi_2, \eta_2) = m(- (\xi_1 + \eta_1), \eta_1, (- \xi_2 + \eta_2), \eta_2)\) and \(m(\xi_1, \eta_1, \xi_2, \eta_2) = m^1(\xi_1, - (\xi_1 + \eta_1), \xi_2, - (\xi_2 + \eta_2))\).

We only choose one case to prove, the remaining cases are the same.

By a change of variables,
\[
\begin{align*}
m^1_{j,k}(\xi_1, \eta_1, \xi_2, \eta_2) &= m(- 2^j (\xi_1 + \eta_1), 2^j \eta_1, 2^j \xi_2, 2^j \eta_2) \\
&\approx m(2^j \xi_1, 2^j \xi_2, 2^j \eta_1, 2^j \eta_2) |\psi_1(- (\xi_1 + \eta_1), \eta_1)\|_{\psi_2(- (\xi_2 + \eta_2), \eta_2)}
\end{align*}
\]
(3.29)
Since \[\sqrt{|\xi + \eta|^2 + |\eta|^2} \approx \sqrt{|\xi|^2 + |\eta|^2},\]
then we can obtain
\[
\begin{align*}
\sup_{j,k \in \mathbb{Z}} \|m_{j,k}^1\|_{H^{1/2}} & \leq \sup_{j,k \in \mathbb{Z}} \|m_{j,k}^2\|_{H^{1/2}}.
\end{align*}
\]
(3.30)
Therefore, we have finished the proof of Theorem 1.7. \(\square\)

**Remark 3.1.** In the proof of Theorem 1.7, we only assume \(p, q > 2n/s, s > n\), it implies that the target space \(L'\) may be the quasi Banach space, where \(r\) depends on \(s\). \(\square\)

### 4. Proof of Theorem 1.9

This section is devoted to establishing the second main theorem of this paper on weighted estimates for the multi-linear and multi-parameter Fourier multipliers with limited smoothness, namely, Theorem 1.9. Before we prove Theorem 1.9, we recall some useful facts about product \(A_p(\mathbb{R}^n \times \mathbb{R}^n)\) weights.

**Lemma 4.1 ([31]).** Let \(1 < p < \infty\) and \(w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)\). Then
\[(1) \quad w^{1/p} \in A_p(\mathbb{R}^n \times \mathbb{R}^n)
\]
(2) there exists \(1 < q < p\) such that \(w \in A_q(\mathbb{R}^n \times \mathbb{R}^n)\).

**Lemma 4.2.** Suppose that \(w_j \in A_p(\mathbb{R}^n \times \mathbb{R}^n)\) with \(1 \leq j \leq m\) for some \(1 \leq p_1, \ldots, p_m \leq \infty\) and let \(0 < \theta_1, \ldots, \theta_m < 1\) be such that \(\theta_1 + \cdots + \theta_m = 1\). Then
\[w_1^{\theta_1} \cdots w_m^{\theta_m} \in A_{\max(p_1, \ldots, p_m)}.
\]
(4.1)

**Proof.** First note that \(w_j \in A_{\max(p_1, \ldots, p_m)}\), for \(j = 1, \ldots, m\), then apply Hölder’s inequality, we can obtain the conclusion.

**Lemma 4.3 ([26]).** Let \(1 < p < \infty\) and \(w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)\). Then there exists a constant \(C > 0\) such that
\[
\left\| \left\{ \sum_{k \in \mathbb{Z}} (M_{\xi} f_k) \right\}^{1/q} \right\|_{L^p(w)} \leq C \left\| \left\{ \sum_{k \in \mathbb{Z}} (f_k) \right\}^{1/q} \right\|_{L^p(w)}
\]
for all sequences \(\{f_k\}_{k \in \mathbb{Z}}\) of locally integrable functions on \(\mathbb{R}^{2n}\).

**Lemma 4.4 ([27]).** Let \(1 < p < \infty\) \(w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)\), and let \(\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)\) be such that \(\text{supp} \psi_1 \subset \{ \xi \in \mathbb{R}^n : 1/a \leq |\xi| \leq a \}\) for some \(a > 1\), \(\text{supp} \psi_2 \subset \{ \xi \in \mathbb{R}^n : 1/b \leq |\xi| \leq b \}\) for some \(b > 1\). Then there exists a constant \(C > 0\) such that
\[
\left\| \left\{ \sum_{k \in \mathbb{Z}} |\psi_1(D/2^k)| \psi_2(D/2^k) f \right\}^{1/2} \right\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \quad \text{for all } f \in L^p_w(\mathbb{R}^n).
\]
(4.3)
Moreover, if \(\sum_{j \in \mathbb{Z}} \psi_3(\xi/2^j) = 1\) for all \(\xi \neq 0\), for \(i = 1, 2\), then
\[
\left\| \left\{ \sum_{k \in \mathbb{Z}} |\psi_i(D/2^k)| \psi_2(D/2^k) f \right\}^{1/2} \right\|_{L^p(w)} \approx \|f\|_{L^p(w)} \quad \text{for all } f \in L^p(\mathbb{R}^n).
\]
(4.4)
Lemma 4.1. If $0 < p < \infty$, $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $f$ is a local integrable function in $H^p_w(\mathbb{R}^n \times \mathbb{R}^n)$. Then
\[
\|f\|_{L^p(w)} \lesssim \left\| \sum_{j,k \in \mathbb{Z}} |\Psi_j(D/2^j)\Psi_k(D/2^k)f|^2 \right\|_{L^p(w)}^{1/2}.
\] (4.5)

We first prove Theorem 1.9 under assumption (i) in Theorem 1.9. Since $2n/s_1 < \min\{2, p\}$ and $w_1 \in A_{p_1/2n}$, by Lemma 4.1, we can take $2n/s_1 < p_1 < \min\{2, p\}$ satisfying $w_1 \in A_{p_1/p_1}$, the same is for $w_2$. Then
\[
\|T_m(f, g)\|_{L^p(w)} \lesssim \sup_{j,k \in \mathbb{Z}} \|m^1_{j,k}\|_{H^{1/2}} \left\| \left\{ \sum_{j,k} (M_\|\psi_j(D)\psi_k(D)f\|)^{2/t} \right\}^{1/2} \right\|_{L^p(w)} \lesssim \sup_{j,k \in \mathbb{Z}} \|m^1_{j,k}\|_{H^{1/2}} \left\| \left\{ \sum_{j,k} (M_\|\psi_j(D)\psi_k(D)f\|)^{2/t} \right\}^{1/2} \right\|_{L^p(w)}.
\] (4.6)

where we take $t = \max\{p_1, q_1\}$, then $w_1 \in A_{p_1/t}$ and $w_2 \in A_{q_1/t}$.

To conclude the weighted estimates for the Fourier multipliers $m$, we need to do estimates corresponding to other symbols. Since the estimates for the remaining symbols in other groups are similar to that of $m$, we omit the details here.

Next, we give the proof of Theorem 1.9 under condition (ii) we consider case $p = \min\{p, q\}$. Since $p' < (2n/s')$, then $\max\{1/r', 1/q\} < 1/r' + 1/q = 1/p < s/2n$, that is, $r', q > 2n/s$. Hence $2n/s < \min\{2, r', q\}$.

Since $1/2 < s/2n \leq 1$ and $w_1^{-1/r'} \in A_{r'/s/(2n)}$, $w_2^{-1/r'} \in A_{r'/s/(2n)}$, by Lemma 4.1 we have
\[
w_1^{-1/r'} \in A_{r'/s/(2n)} \subset A_{r'}, \quad \text{then} \quad w_1 \in A_r
\]
(4.7)
\[
w_2^{-1/r'} \in A_{r'/s/(2n)} \subset A_{r'}, \quad \text{then} \quad w_2 \in A_r
\]
(4.8)
\[
w_1^{-1/r'} = w_1^{-1/r'} r' \quad w_2^{-1/r'} = 2 w_2^{-1/r'} \in A_{r'/s/(2n)}
\]
(4.9)
where (4.9) is from Lemma 4.2.

It is from the assumption that $p \leq q$, we also have $r \leq q/2$, then $w_2 \in A_r \subset A_{q/2} \subset A_{q/2}$. Since $w_1^{-1/r'} \in A_{r'/s/(2n)}$, $w_2 \in A_r \subset A_{q/2}$, by Lemma 4.2 we can take $2n/s < t < \min\{2, r', q\}$ such that
\[
w_1^{-1/r'} \in A_{r'/t}, \quad w_2 \in A_{q/t}.
\]
(4.10)

By duality and (3.29), it is enough to prove
\[
\|T_m^{1,1} \|_{L^{p'}(w_1^{-1/r'}, w_2^{1/r'}) \rightarrow L^{p'}(w_1^{-1/r'}, w_2^{1/r'})} \leq C \sup_{j,k \in \mathbb{Z}} \|m^1_{j,k}\|_{H^{1/2}}.
\]
(4.11)

From the proof of Theorem 1.7, we have
\[
\|T_m^{1,1} (f, g)\|_{L^{p'}(w_1^{-1/r'}, w_2^{1/r'})} \lesssim \left\| \sum_{j,k \in \mathbb{Z}} |\Psi_j(D/2^j)\Psi_k(D/2^k)f|^2 \right\|_{L^{p'}(w_1^{-1/r'}, w_2^{1/r'})}^{1/2} \lesssim \sup_{j,k \in \mathbb{Z}} \|m^1_{j,k}\|_{H^{1/2}} \left\| \left\{ \sum_{j,k} (M_\|\psi_j(D)\psi_k(D)f\|)^{2/t} \right\}^{1/2} \right\|_{L^{p'}(w_1^{-1/r'}, w_2^{1/r'})}.
\]
\[ \sup_{j,k} \left\| m_{j,k}^1 \right\|_{H^{1/2}} \left\| \sum_j \left( \left( \tilde{\psi}_k(D) \tilde{\psi}_l(D) f^l \right) \right)(x_1, x_2) \right\|^{1/2} \left\| f \right\|_{L^p(w)} \right\|_{L^2(w)} \]

(4.12)

The weighted estimates for the Fourier multiplier operators corresponding to the remaining symbols are the same as with \( T_{m1} \), thus we finish the proof of Theorem 1.9.

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