1. (a) Let $P$ be Lebesgue measure on $[0,1]$. For any function $F$ on $\mathbb{R}$ that satisfies the conditions of Theorem (1.3) (1)-(3) for a distribution function, let

$$G(y) = \sup \{ x : F(x) \leq y \}.$$  

Then $G$ is a r.v. (i.e. measurable) on $[0,1]$ which has distribution function $F$. Thus without the Hopf Extension Theorem one can prove that every distribution function is the distribution function of a probability measure.

(b) Let $X$ be a random variable whose distribution function $F(x)$ is continuous. Then $F(X)$ has distribution measure the Lebesgue measure on $[0,1]$ (i.e., the uniform distribution on $[0,1]$).

2. For any collection $\{X_\alpha\}_{\alpha \in A}$ of jointly distributed random variables, let $\mathcal{F}(\{X_\alpha\})$ denote the smallest $\sigma$-algebra with respect to which all of the $X_\alpha$ are measurable.

(a) For any random variable $X$, show that a random variable $Y$ is $\mathcal{F}(X)$-measurable if and only if there is a Borel-measurable function $f$ such that $Y = f(X)$.

(b) Extend part (a) to a finite collection $\{X_1, \ldots, X_n\}$ of random variables.

3. Let $X$ be a random variable.

(a) For any $p > 0$, $E(|X|^p) < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{p-1} P[|X| \geq n] < \infty.$$  

(b) For any $p > 0$, if $E(|X|^p) < \infty$, then

$$\lim_{x \to \infty} x^p P[|X| > x] = 0.$$  

(\ast)

Conversely, if (\ast) holds, then $E(|X|^{p-\varepsilon}) < \infty$ for all $0 < \varepsilon < p$.

4. Let $X$ be a random vector with values in $\mathbb{R}^d$. We say that $X$ has density $f(t)$ ($t \in \mathbb{R}^d$) if for every Borel set $E$ in $\mathbb{R}^d$,

$$P[X \in E] = \int_E f(t) dt,$$

where $dt$ denotes Lebesgue measure.

Continued...
(a) If $X$ has density $f$, show that:

1. $f(t) \geq 0$ a.s.
2. $\int_{\mathbb{R}^d} f(t) \, dt = 1$.

(b) Let $d = 1$. and $X$ have distribution function $F(x)$. Prove:

1. If $f$ is continuous at $t_0$, then $F'(t_0) = f(t_0)$.
2. If the distribution function $F$ of $X$ is differentiable on $\mathbb{R}$, prove that such an $X$ has a density. (Hint: First prove that $F$ is absolutely continuous. Then use the Fundamental Theorem of Calculus for Lebesgue integrals.)

(c) If $X$ and $Y$ are independent random vectors values in $\mathbb{R}^d$ and densities $f$ and $g$, respectively, then $X + Y$ has density

$$h(t) = \int_{\mathbb{R}^d} f(t - s)g(s) \, ds \text{ a.s.}$$

(Hint: Why is the integrand above measurable as a function on $\mathbb{R}^{2d}$?)

5. For any collection $Y$ and $\{X_\alpha\}_{\alpha \in A}$ of jointly distributed random variables, define

$$E(Y \mid \{X_\alpha\}) = E(Y \mid \mathcal{F}(\{X_\alpha\}).$$

Let $X = (X_1, X_2)$ be a random vector having (joint) density $f(t) \ (t = (t_1, t_2))$ with $E(\|X\|) < \infty$.

(a) Show that each $X_i, \ i = 1, 2$, has density $f_{X_i}(t_i) = \int_{\mathbb{R}} f(t_1, t_2) \, dt_{3-i}$ a.s..

(b) Thinking of $X$ as a random vector on $\mathbb{R}^2$, show that

$$E(X_2 \mid X_1)(t_1, t_2) = \frac{1}{f_{X_1}(t_1)} \int_{\mathbb{R}} u f(t_1, u) \, du.$$

(c) Suppose that $X$ has the bivariate normal distribution with density

$$f(t) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \int_E \exp \left[ -\frac{1}{2(1 - \rho^2)} \left( \frac{t_1^2}{\sigma_1^2} - \frac{2\rho t_1 t_2}{\sigma_1 \sigma_2} + \frac{t_2^2}{\sigma_2^2} \right) \right] \, dt,$$

where $\sigma_1 > 0, \sigma_2 > 0$, and $0 < \rho < 1$. Calculate $E(X_2 \mid X_1)$.

6. An atom in a probability space $(\Omega, \mathcal{F}, P)$ is a set $A \in \mathcal{F}$ such that $P(A) > 0$ and for every $E \in \mathcal{F}$ with $E \subset A$, either $P(E) = P(A)$ or $P(E) = 0$. Suppose that $P$ has no atoms. For any $0 < \alpha < 1$, prove that there is a set $E \in \mathcal{F}$ such that $P(E) = \alpha$. 