Throughout this assignment, $X$ is a Hilbert space over $\mathbb{C}$.

1. (a) Let $x_n, y_n, n = 1, 2, \ldots$ be elements of the closed unit ball of $X$. If $\lim_{n \to \infty} \langle x_n, y_n \rangle = 1$, prove that $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

(b) If $x_n \in X$, $n = 1, 2, \ldots$, $x_n \to x$ weakly, and $\|x_n\| \to \|x\|$ as $n \to \infty$, prove that $x_n \to x$ in norm.

2. Let $X$ be separable with c.o.n.s. $\{e_1, e_2, \ldots\}$. For $\varepsilon_n > 0$, $n = 1, 2, \ldots$, let

$$K = \{x \in X : |\langle x, e_n \rangle| \leq \varepsilon_n, n = 1, 2, \ldots\}$$

Prove that $K$ is compact in $X$ if and only if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. (Sets of this type are called Hilbert cubes.)

3. Let $T \in \mathcal{B}(X)$.

(a) Prove: The operator $T$ is normal if and only if $\|T^*x\| = \|Tx\|, x \in X$.

(b) Let $T = W|T|$ be the polar decomposition of $T$. Prove:

(i) If $T$ is normal then $T = |T|W$.

(ii) If $T = |T|W$ and $T$ is invertible, then $T$ is normal.

(c) Suppose $T$ is self-adjoint. Prove that the series

$$\exp(iT) = I + iT + \frac{(iT)^2}{2!} + \frac{(iT)^3}{3!} + \cdots$$

converges in $\mathcal{B}(X)$ to a unitary operator.

4. Let $S$ and $T$ be normal operators on $X$.

(a) If $A$ is an invertible operator on $X$ and $T$ commutes with $A^*A$, then $ATA^{-1}$ is normal.

(b) If $S$ and $T^*$ commute, prove that $S + T$ and $ST$ are normal.

(c) Find positive operators $S$ and $T$ such that $ST$ is not normal. (Hint: Of course, $S$ and $T$ cannot commute. You can find such operators on $\mathbb{C}^2$.)
5. (a) If \( T \in \mathcal{B}(X) \) with \( \|T\| < 1 \), then \( I - T \) is invertible in \( \mathcal{B}(X) \). \( \text{(Hint: Show that the series } I + T + T^2 + \cdots \text{ converges in norm to } (I - T)^{-1}. \) \)

(b) The set of invertible elements in \( \mathcal{B}(X) \) is an open group. \( \text{(Hint: If } T \text{ is invertible, then for any } S \in \mathcal{B}(X), \)
\[
\|I - T^{-1}S\| = \|T^{-1}(T - S)\| \leq \|T^{-1}\| \|T - S\|. \]

6. For \( T \in \mathcal{B}(X) \), the spectrum of \( T \) is the set
\[
\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } \mathcal{B}(X) \}
\]
(Recall that when \( \dim X < \infty \) then \( \sigma(T) \) is the set of eigenvalues of \( T \).) It can be shown that \( \sigma(T) \neq \emptyset \) for all \( T \). Prove each of the following assertions for an operator \( T \) on \( X \).

(a) \( \sigma(T^*) = \{ \overline{\lambda} : \lambda \in \sigma(T) \} \). If \( T \) is self adjoint, then \( \sigma(T) \subset \mathbb{R} \).

(b) If \( T \) is invertible then
\[
\sigma(T^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(T) \right\}.
\]

(c) If \( \lambda \in \sigma(T) \), then \( |\lambda| \leq \|T\| \); hence \( \sigma(T) \) is a nonempty compact subset of \( \mathbb{C} \). \( \text{(Hint: If } |\lambda| > \|T\|, \text{ then } \|\lambda^{-1}T\| < 1. \) \)

(d) If \( U \) is a unitary operator, then \( \sigma(U) \subset \mathbb{T} = \{ z : |z| = 1 \} \).