A gambler plays a game in which she either wins one dollar with probability $p$ or loses one dollar with probability $1-p$. Assume she quits when she either goes broke or attains a fortune of $F$ dollars. If $X_n$ denotes her fortune after $n$ plays, then $\{X_n, n \geq 0\}$ is a Markov chain with transition probabilities

$$p_{00} = 1; \quad p_{i,i+1} = p, \quad p_{i,i-1} = 1 - p, \quad i = 1, 2, \ldots, F - 1; \quad p_{FF} = 1; \quad p_{ij} = 0 \text{ otherwise.}$$

States 0 and $F$ are called absorbing states.

Suppose $F = 5$ dollars. If she starts out with 2 dollars, let us determine the probability that she will be broke within 4 plays. Using the Chapman-Kolmogorov equations, we need to introduce the transition matrix $P$, calculate $P^4$, and then read off $p_{20}^{(4)}$. This was done using mathematical symbolic manipulation software, but it could be done on a graphing calculator.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 - p & 0 & p & 0 & 0 & 0 \\ 0 & 1 - p & 0 & p & 0 & 0 \\ 0 & 0 & 1 - p & 0 & p & 0 \\ 0 & 0 & 0 & 1 - p & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 - p & p(1 - p) & 0 & p^2 & 0 & 0 \\ (1 - p)^2 & 0 & 2p(1 - p) & 0 & p^2 & 0 \\ 0 & (1 - p)^2 & 0 & 2p(1 - p) & 0 & p^2 \\ 0 & 0 & (1 - p)^2 & 0 & p(1 - p) & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ (1 - p)(1 + p - p^2) & 2p^2(1 - p) & 0 & 3p^3(1 - p) & 0 & p^4 \\ (1 - p)^2(1 + 2p - 2p^2) & 0 & 5p^2(1 - p)^2 & 0 & 3p^3(1 - p) & p^3 \\ (1 - p)^3 & 3p(1 - p)^3 & 0 & 5p^2(1 - p)^2 & 0 & p^2(1 + 2p - 2p^2) \\ (1 - p)^4 & 0 & 3p(1 - p)^3 & 0 & 2p^2(1 - p)^2 & p(1 + p - p^2) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In particular, we obtain $p_{20}^{(4)} = (1 - p)^2(1 + 2p - 2p^2)$. 