MATHEMATICS 5070

ASSIGNMENT 6

1. Suppose $f \in \mathcal{R}(a, b)$ and $g$ is a function on $[a, b]$ that agrees with $f$ except at a finite number of points, i.e., \( \{x : f(x) \neq g(x)\} \) is finite. Prove that $g \in \mathcal{R}(a, b)$ and $\int_a^b f = \int_a^b g$.

2. A function $f$ on a domain $D$ is called a Lipschitz function if there is a number $K > 0$ such that for all $x, y \in D$,

$$ |f(x) - f(y)| \leq K|x - y|. $$

Clearly every Lipschitz function on $D$ is uniformly continuous. Prove each of the following.

(a) Let $f \in \mathcal{R}(a, b)$ with $m \leq f(x) \leq M$ for all $a \leq x \leq b$, and let $\varphi : [m, M] \to \mathbb{R}$ be a Lipschitz function. The $\varphi \circ f \in \mathcal{R}(a, b)$.

(b) If $f$ is differentiable on $(a, b)$ ($-\infty \leq a < b \leq \infty$) and $f'$ is bounded on $(a, b)$, then $f$ is a Lipschitz function on $(a, b)$.

(c) If $f \in \mathcal{R}(a, b)$ then $f^n \in \mathcal{R}(a, b)$, $n = 0, 1, 2, \ldots$.

(d) Let $f, g \in \mathcal{R}(a, b)$. Then $fg \in \mathcal{R}(a, b)$. (Hint: $2xy = (x + y)^2 - x^2 - y^2$.)

3. Consider the Dirichlet function on $[0, 1]$,

$$ f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms, } p, q \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}. $$

Prove that $f \in \mathcal{R}(0, 1)$ and $\int_0^1 f = 0$.

4. Suppose $a_n > 0$, $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} a_n < \infty$. Let $\{x_n\}$ be a finite or infinite sequence of distinct points in $[0, 1]$. Define the function $f$ on $[0, 1]$ by

$$ f(x) = \sum_{\{n: x_n \leq x\}} a_n. $$

(In general, $f(x)$ is defined by an infinite series of positive numbers. There is no restriction on the location or order of the points $x_n$; for instance, the sequence $\{x_n\}$ could be an enumeration of the rational points in $[0, 1]$.)

(a) prove that $f$ is a monotone increasing function on $[0, 1]$.

(b) Prove that $f$ is discontinuous at each $x_n$ and continuous at every other $x$ in $[0, 1]$.

(We know that every monotone function has at most a countable set of points of discontinuity. This problem shows that one can stipulate exactly those points of discontinuity.)
(c) Can you calculate $\int_0^1 f$? If you cannot do it for every sequence $\{x_n\}$, for which such sequences can you do it?

5. Let $f$ be continuous on $\mathbb{R}$ and $g$ and $h$ be differentiable on $\mathbb{R}$. Prove that

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = f(h(x))h'(x) - f(g(x))g'(x).$$