MATHEMATICS 5070

ASSIGNMENT 2

1. (a) Prove that the sequence \( a_n = \frac{1 + 2 + 3 + \cdots + n}{n^2}, \ n = 1, 2, \ldots \) converges to 1/2.

(b) Let \( a_n = \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}, \ n = 1, 2, \ldots \). What is the limit of the sequence \( \{a_n\} \)? Prove it.

2. Suppose that the sequence \( \{a_n\} \) converges to \( A \). Let \( b_n = \frac{a_n + a_{n+1}}{2}, \ n = 1, 2, \ldots \)

Does the sequence \( \{b_n\} \) converge? If so, find the limit and prove that \( \{b_n\} \) converges to that limit. Otherwise, give an example of a sequence \( \{a_n\} \) such that the derived sequence \( \{b_n\} \) diverges.

3. Let \( 0 \leq \alpha \leq \beta \), and define \( \{a_n\} \) by \( a_n = (\alpha^n + \beta^n)^{1/n} \). Prove that \( \lim_{n \to \infty} a_n = \beta \).

4. (a) Suppose that \( \{a_n\} \) and \( \{b_n\} \) are sequences of positive numbers such that

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0.
\]

Prove that \( \lim_{n \to \infty} a_n = \infty \) if and only if \( \lim_{n \to \infty} b_n = \infty \).

(b) Show that the conclusion of part (a) fails if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) or \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \).

5. (a) Prove that the sequence \( \{a_n\} \) with

\[
a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots 2n}
\]

converges to some \( A \) such that \( 0 \leq A < 1/2 \).

(b) Prove that the sequence \( \{b_n\} \) with

\[
b_n = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n + 1)}
\]

converges to some \( B \) such that \( 0 \leq B < 2/3 \).

(c) Prove that the sequence \( \{a_n b_n\} \) converges to 0.

(d) Conclude from (a)–(c) that \( A = 0 \).